THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 68 NUMBER 3

DETROIT PUBLIC LIBRARY
MAR 20 1961
TECHNOLOGY DEPARTMENT

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MARCH 1961

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $8.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July.
August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

THE FOURIER TRANSFORM AND MEAN CONVERGENCE

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At the undergraduate level the Fourier transform is customarily studied by pointwise convergence, and only absolutely integrable functions are considered. In view of the importance of convergence in the mean and of the invariance of the integral of the absolute square under the Fourier transform, it would seem desirable to have a treatment of the Fourier transform in terms of convergence in the mean, still using only Riemann integrals, since ordinarily no other integral is known to undergraduates. The present paper presents such a treatment of two fundamental theorems, the invariance of the integral of the absolute square and the inversion theorem.

Let us say that a (complex-valued) function f on the real axis is *square-integrable* if f is (Riemann) integrable over every interval and $|f|^2$ is integrable over the whole real axis. The Fourier theory makes no distinction between functions f_1 and f_2 which are equal at so many places that the integral of $|f_1-f_2|^2$ is 0 (or, what amounts to the same thing, the integral of $|f_1-f_2|$ is 0). Hence such functions will be called *equivalent*. Properly speaking, the range and the domain of the transform are not functions, but equivalence classes of functions. But we follow custom in saying that the transformation maps functions on functions, and we "identify" any two equivalent functions.

If f is square-integrable we define its norm to be

$$\|f\| = \left[\int |f(t)|^2 dt\right]^{1/2}.$$

(When the limits of integration are omitted they are understood to be $-\infty$ and ∞.) We assume it known that if f and g are square-integrable so is $f+g$, and $\|f+g\| \leqq \|f\| + \|g\|$. It is also convenient, though by no means necessary, to make use of the inner product

$$(f, g) = \int f(t)\overline{g(t)}\, dt,$$

along with the most elementary properties of the inner product.

If f is square-integrable, for each positive number A we define $g_{[A]}$ to be the function for which

$$g_{[A]}(s) = (2\pi)^{-1/2} \int_{-A}^{A} e^{-ist} f(t)\, dt \qquad (-\infty < s < \infty). \tag{1}$$

If there is a function g, necessarily square-integrable, such that

$$\lim_{A\to\infty} \|g_{[A]} - g\| = 0,$$

we say that g is the Fourier transform of f (although it might be better to call it the Fourier-Plancherel transform), and we write $g = Tf$. The conjugate Fourier transform $\overline{T}f$ is similarly defined, with e^{ist} in place of e^{-ist}. The connection between this convergence in mean and the pointwise convergence of $g_{[A]}$ to a limit is not easily studied with only the Riemann integral at hand, but the next lemma states a simple and useful relationship.

LEMMA 1. *Let $h_1, h_2, h_3, \cdots$ be a Cauchy sequence of square-integrable functions (that is, assume that $\|h_m - h_n\|$ tends to 0 as m and n tend to ∞.) Let h_0 be defined on the real axis, and let B be a set of real numbers such that no interval contains more than a finite number of points of B. Assume that whenever $[a, b]$ is a closed interval containing no point of B, $h_n(t)$ converges uniformly to $h_0(t)$ on $[a, b]$. Then h_0 is square-integrable, and $\lim_{n\to\infty} \|h_n - h_0\| = 0$.*

Proof. Let A be positive, and let $b_1, \cdots, b_p$ be the points of B between $-A$ and A, in increasing order. If ϵ is positive, there is an n_ϵ such that if m and n both exceed n_ϵ, $\|h_n - h_m\| < \epsilon/2$. Hence for any small positive δ

$$\int_{-A+\delta}^{b_1-\delta} + \int_{b_1+\delta}^{b_2-\delta} + \cdots + \int_{b_p+\delta}^{A-\delta} |h_n(t) - h_m(t)|^2 dt \leqq (\epsilon/2)^2.$$

On each interval of integration $h_m(t)$ tends uniformly to $h_0(t)$, so if we fix n at any integer above n_ϵ we have the same inequality with h_0 in place of h_m. Letting δ tend to 0 gives

$$\int_{-A}^{A} |h_n(t) - h_0(t)|^2 dt \leqq (\epsilon/2)^2,$$

and letting A tend to ∞ yields $\|h_n - h_0\|^2 \leqq (\epsilon/2)^2 < \epsilon^2$. This establishes both conclusions.

For any two real numbers a, b $(b > a)$ we define $f_{a,b}$ to be

$$f_{a,b}(t) = [\operatorname{sgn}(t - a) - \operatorname{sgn}(t - b)]/2;$$

that is,

$$f_{a,b}(t) = \begin{cases} 0 & \text{if } t < a \text{ or } t > b, \\ 1/2 & \text{if } t = a \text{ or } t = b, \\ 1 & \text{if } a < t < b. \end{cases} \tag{2}$$

It is a trivial calculation that its Fourier transform $g_{a,b}$ is

$$g_{a,b}(s) = \frac{i}{\sqrt{(2\pi)}} \frac{e^{-isb} - e^{-isa}}{s} \qquad (s \neq 0). \tag{3}$$

(This has the finite limit $(b-a)/\sqrt{(2\pi)}$ as s tends to 0.) These functions are tools in proving both our theorems.

LEMMA 2. *If f is square-integrable and has a Fourier transform g, then* $\|g\|^2 = \lambda\|f\|^2$, *where*

$$\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2}\, dv.$$

Proof. If $[a, b]$ and $[c, d]$ are any two intervals, we compute

$$(g_{a,b}, g_{c,d}) = (2\pi)^{-1}\int s^{-2}[e^{-is(b-d)} + e^{-is(a-c)} - e^{-is(b-c)} - e^{-is(a-d)}]ds.$$

When we replace the imaginary exponentials by means of Euler's formula we find that the imaginary part of the integrand is an odd function, so its integral from $-\infty$ to ∞ is 0, and the inner product is real. In particular, if $c=a$ and $d=b$, by a trigonometric identity and the change of variable of integration $v=s(b-a)/2$, we obtain

$$\|g_{a,b}\|^2 = \pi^{-1}\int s^{-2}[1 - \cos s(b-a)]ds = \lambda(b-a).$$

If $a \leqq b \leqq c$, then $f_{a,c}=f_{a,b}+f_{b,c}$, and by the linearity of the transformation $g_{a,c}=g_{a,b}+g_{b,c}$. Moreover, because the inner product $(g_{a,b}, g_{b,c})$ is real it is equal to $(g_{b,c}, g_{a,b})$. Hence

$$\begin{aligned}\|g_{a,c}\|^2 &= (g_{a,b} + g_{b,c}, g_{a,b} + g_{b,c}) \\ &= \|g_{a,b}\|^2 + 2(g_{a,b}, g_{b,c}) + \|g_{b,c}\|^2.\end{aligned}$$

For three of these terms we substitute their values as just computed. The left member then eliminates the first and last terms of the right member, and we have $(g_{a,b}, g_{b,c})=0$. Finally, if $a \leqq b \leqq c \leqq d$, by the proof just completed both $(g_{a,b}, g_{b,c})$ and $(g_{a,b}, g_{b,d})$ vanish, hence so does their difference $(g_{a,b}, g_{c,d})$.

Next let f be a step-function, with values $c_1, \cdots, c_p$ on the respective intervals (in increasing order) $t_0 < t \leqq t_1, \cdots, t_{p-1} < t \leqq t_p$. For notational simplicity we write f_j to stand for f_{t_{j-1},t_j}; then (except perhaps at the points $t_0, \cdots, t_p$) $f=c_1f_1+\cdots+c_pf_p$. Hence, if $g_j=Tf_j$ and $g=Tf$, we have $g=c_1g_1+\cdots+c_pg_p$. Multiplying this by its conjugate and integrating yields, in view of the fact that $(g_j, g_k)=0$ if $j \neq k$, and $(g_j, g_j)=\lambda(t_{j-1}-t_j)$,

$$\begin{aligned}\|g\|^2 &= \lambda\,|\,c_1\,|^2(t_1 - t_0) + \cdots + \lambda\,|\,c_p\,|^2(t_p - t_{p-1}) \\ &= \lambda\int |\,f(t)\,|^2 dt = \lambda\|f\|^2.\end{aligned}$$

Thus the conclusion of the lemma holds when f is any step-function.

Next let f be Riemann integrable (therefore bounded) on an interval $[-K, K]$ and vanish outside that interval; let M be an upper bound for $|f|$. By definition of the Riemann integral there is a sequence of step-functions

$S_1, S_2, \cdots$ vanishing outside $[-K, K]$, with absolute values not greater than M, and such that

$$\lim_{n\to\infty} \int_{-K}^{K} |S_n(t) - f(t)|\, dt = 0.$$

Since $|S_n - f|^2 \leqq 2M(|S_n - f|)$ this implies that $\|S_n - f\|$ tends to 0. Hence $\|S_n - S_m\|$ tends to 0 as m and n increase. If we write $G_n = TS_n$, $g = Tf$, by the preceding paragraph $\|G_n - G_m\|$ also tends to 0 as m and n increase. Also the limiting process defining g is trivial; $g_{[A]}$ has the same value for all A greater than K. For all s,

$$|g(s) - G_n(s)| \leqq (2\pi)^{-1/2} \int_{-K}^{K} |e^{-ist}| \cdot |f(t) - S_n(t)|\, dt,$$

and as n increases this last tends to 0 and is independent of s. Hence $G_n(s)$ tends to $g(s)$ uniformly for all s. By Lemma 1, g is square-integrable and $\|g - G_n\|$ tends to 0. But then by the triangle inequality and the preceding paragraph

$$\|g\|^2 = \lim_{n\to\infty} \|G_n\|^2 = \lim_{n\to\infty} \lambda \|S_n\|^2 = \lambda \|f\|^2.$$

Finally let f be any square-integrable function that has a Fourier transform $g = Tf$, and let A be any positive number. Define $f_A(t)$ to be $f(t)$ if $-A \leqq t \leqq A$, 0 otherwise. Then, with the notation of the definition $g_{[A]} = Tf_A$, and by the preceding paragraph $\|g_{[A]}\|^2 = \lambda \|f_A\|^2$. As A increases, by definition of Tf we have $\lim \|g_{[A]} - g\|^2 = 0$, so that $\|g_{[A]}\|^2$ tends to $\|g\|^2$; and by definition of the integral from $-\infty$ to ∞, $\|f_A\|^2$ tends to $\|f\|^2$. Hence $\|g\|^2 = \lambda \|f\|^2$, completing the proof of the lemma.

COROLLARY 1. *Let f be square-integrable. If there is a point set B such that only finitely many points of B lie in any interval of the real axis, and the functions $g_{[A]}(s)$ of equation* (1) *converge uniformly to a limit $g(s)$ on every closed interval that contains no point of B, then g is square-integrable, and is the Fourier transform of f.*

Proof. Use the notation of the last paragraph of the preceding proof. If A goes through any sequence of values tending to ∞, $\|f_A - f\|$ tends to 0, so the f_A form a Cauchy sequence. By Lemma 2, so do their transforms $g_{[A]}$. The conclusion follows by way of Lemma 1.

COROLLARY 2. *If f is square-integrable and also is absolutely integrable, the transforms $g_{[A]}(s)$ converge uniformly to a continuous limit $g(s)$, and g is the Fourier transform of f.*

Proof. For all s,

$$|g(s) - g_{[A]}(s)| \leqq (2\pi)^{-1/2} \left\{ \int_{-\infty}^{\infty} - \int_{-A}^{A} |e^{-ist}| \cdot |f(t)|\, dt \right\} \cdot$$

This is independent of s and tends to 0 as A increases, so $g_{[A]}(s)$ converges uniformly to $g(s)$; since $g_{[A]}$ is continuous, so is g. By Corollary 1, g is square-integrable and $g=Tf$.

If in particular we take $f(t)$ to be 0 if $t<0$ and to be e^{-t} if $t\leqq 0$, by Corollary 2 its Fourier transform g satisfies $g(s)=(2\pi)^{-1/2}(1+is)^{-1}$, as we compute without trouble. Then

$$\|f\|^2 = \int_0^\infty e^{-2t}dt = 1/2,$$

$$\|g\|^2 = (1/2\pi)\int_{-\infty}^{\infty}(1+s^2)^{-1}ds = 1/2,$$

and so by Lemma 2 we have $\lambda=1$. Now Lemma 2 becomes

THEOREM 1. *If f is square-integrable and has a Fourier transform, then* $\|Tf\|^2 = \|f\|^2$.

If in (1) we had replaced e^{-ist} by e^{ist} the effect would have been only to replace $g_{[A]}$ by its reflection $\rho g_{[A]}$, for which $\rho g_{[A]}(s)=g_{[A]}(-s)$. This has no effect on the norm, so f has a Fourier transform Tf if and only if it has a conjugate Fourier transform $\overline{T}f$, and in that case $\|\overline{T}f\|=\|Tf\|=\|f\|$.

Our inversion theorem asks nothing more than that f have a Fourier transform.

THEOREM 2. *Let f be square-integrable and have a Fourier transform $g=Tf$. Then g has a conjugate Fourier transform, and $\overline{T}g=f$.*

Proof. For positive A we define, analogously to (1),

$$h_{[A]}(t) = (2\pi)^{-1/2}\int_{-A}^{A} e^{its}g_{a,b}(s)ds. \tag{4}$$

Consider first the function $f_{a,b}$ defined in (2) and its transform $g_{a,b}$ defined in (3). For this, (4) becomes

$$h_{[A]}(t) = (i/2\pi)\int_{-A}^{A}\left[e^{is(t-b)} - e^{is(t-a)}\right]s^{-1}ds$$

$$= (1/2\pi)\int_{-A}^{A}\left[\frac{\sin s(t-a)}{s} - \frac{\sin s(t-b)}{s}\right]ds;$$

the cosine terms have dropped out because $[\cos s(t-a)-\cos s(t-b)]/s$ is odd and its integral from $-A$ to A is 0. By a change of variable in each of the two terms of the right member this becomes

$$h_{[A]}(t) = (1/2\pi)\left[\int_{-A(t-a)}^{A(t-a)}\frac{\sin v}{v}\,dv - \int_{-A(t-b)}^{A(t-b)}\frac{\sin v}{v}\,dv\right].$$

By the usual elementary discussion of its graph we show that $(\sin v)/v$ is integrable from 0 to ∞, hence from $-\infty$ to ∞. Let the latter integral be μ; then for each positive ϵ there is an X_ϵ such that if $X > X_\epsilon$ then

$$\left| \int_{-X}^{X} \frac{\sin v}{v} dv - \mu \right| < \epsilon.$$

If we allow negative X also, this yields

$$\left| \int_{-X}^{X} \frac{\sin v}{v} dv - \mu \operatorname{sgn} X \right| < \epsilon \qquad (|X| > X_\epsilon).$$

Let $[c, d]$ be a closed interval not containing either a or b, and let δ be the (positive) distance from the nearer of these two points to the closed interval $[c, d]$. Then for all A such that $A > X_\epsilon/\delta$ we have $|A(t-a)|$ and $|A(t-b)|$ both greater than X_ϵ for all t in $[c, d]$. The preceding inequality and the last equation for $h_{[A]}$ yield

$$|h_{[A]}(t) - \mu[\operatorname{sgn}(t-a) - \operatorname{sgn}(t-b)]/2\pi| < 2\epsilon \qquad (c \leqq t \leqq d).$$

That is (cf. (2)) $h_{[A]}(t)$ tends to $\mu f_{a,b}(t)/\pi$ uniformly on the interval $[c, d]$. Corollary 1 has an obvious dual for the conjugate Fourier transform; this, with B the set consisting of the two points a and b, informs us that $\mu f_{a,b}/\pi$ is the conjugate Fourier transform of $g_{a,b}$. It is easily seen that μ is positive, so that $\|\mu f_{a,b}/\pi\| = (\mu/\pi)\|f_{a,b}\|$. By the dual of Theorem 1 for the conjugate Fourier transform we have $\|\mu f_{a,b}/\pi\| = \|g_{a,b}\|$, while by Theorem 1 applied to $f_{a,b}$ we have $\|g_{a,b}\| = \|f_{a,b}\|$. These together imply that $\mu/\pi = 1$, and so $f_{a,b}$ is the conjugate Fourier transform of $g_{a,b}$.

By the linearity of the transformation, it follows at once that $\overline{T}(TS) = S$ for every step-function S.

Now let ϵ be any positive number. We can find a number K such that

$$0 \leqq \|f\| - \left[\int_{-K}^{K} |f(t)|^2 dt \right]^{1/2} < \epsilon/6;$$

then (as in the proof of Lemma 2) we can find a step-function S vanishing outside the interval $[-K, K]$ such that

$$\left[\int_{-K}^{K} |S(t) - f(t)|^2 dt \right]^{1/2} < \epsilon/6.$$

These together imply $\|S-f\| < \epsilon/3$. We take Fourier transforms and apply Theorem 1; the result is $\|TS-g\| = \|S-f\| < \epsilon/3$.

Let us define g_A to be the function with values

$$g_A(s) = 0 \quad \text{if } s < -A \text{ or } s > A;$$
$$= g(s) \quad \text{if } -A \leqq s \leqq A.$$

Its conjugate Fourier transform is, as in (4), $h_{[A]}$. There is an A_ϵ such that $\|g_A - g\| < \epsilon/3$ whenever $A > A_\epsilon$, by definition of the integral. From this and the preceding paragraph, $\|g_A - TS\| < 2\epsilon/3$ whenever $A > A_\epsilon$. We take conjugate Fourier transforms and use Theorem 1 and the remark after it to obtain $\|\overline{T}g_A - \overline{T}(TS)\| < 2\epsilon/3$. But since S is a step-function, $\overline{T}(TS) = S$, and by definition $\overline{T}g_A$ is $h_{[A]}$; so we have $\|h_{[A]} - S\| < 2\epsilon/3$. Since $\|S - f\| < \epsilon/3$, it follows that $\|h_{[A]} - f\| < \epsilon$ whenever $A > A_\epsilon$. That is, by definition $\overline{T}g = f$.

A MODEL OF QUASI-EUCLIDEAN SPACE

A. R. AMIR-MOÉZ, Purdue University, AND A. L. FASS, Queens College

A quasi-unitary space is a linear space over the complex field with a quasi-inner product defined for it. In case the field is the field of real numbers we call the space quasi-Euclidean. Before we give the model we shall discuss a simple form of quasi-inner product and quasi-product of matrices. We also discuss a way of estimating sum of squares of some elements of a matrix minus the sum of squares of other elements by defining and using quasi-singular values of a matrix.

1. Definitions and notations. Let V be an n-dimensional unitary space, with an orthonormal base $\{\alpha_1, \cdots, \alpha_n\}$. Let $\xi = (x_1, \cdots, x_n)$ and $\eta = (y_1, \cdots, y_n)$ be two n-tuples of complex numbers, so that $\xi, \eta \in V$. Then we define

$$(\xi, \eta)_k = -\sum_{i=1}^{k} x_i \bar{y}_i + \sum_{i=k+1}^{n} x_i \bar{y}_i$$

to be the quasi-inner product of order k of ξ and η. For the general case see [4], page 150.

Now let $A = (a_{ij})$ and $B = (b_{ij})$ be two n-by-n matrices with complex elements. We define the quasi-product of order k of A and B, written $(AB)_k$, to be the matrix $C = (c_{lm})$ where c_{lm} is the quasi-inner product of the lth row of A and the conjugate of the mth column of B.

Let J_k be a diagonal matrix with -1 for its first k diagonal elements and $+1$ for the rest. Then it is easy to see that

$$(\xi, \eta)_k = (J_k\xi, \eta) = (\xi, J_k\eta),$$

and $(AB)_k = AJ_kB$.

A set of n vectors $\{\beta_1, \cdots, \beta_n\}$ will be called a quasi-orthonormal base of order k if

$$(\beta_i, J_k\beta_i) = -1, \ i = 1, \cdots, k, \qquad (\beta_i, J_k\beta_i) = 1, \ i = k+1, \cdots, n,$$
$$(\beta_i, J_k\beta_j) = 0, \ i \neq j.$$

If the rows of a matrix A form a quasi-orthonormal base of order k, the matrix will be called quasi-unitary of order k, denoted by U_k. The columns of a quasi-unitary matrix form also a quasi-orthonormal base of order k [2].

2. Quasi-orthogonal sets. Let $\{\xi_1, \cdots, \xi_n\}$ be a linearly independent set of vectors, with $(\xi_i, \xi_j)_k = 0$, $i \neq j$. Then the set may be ordered in such a way that

$$(\xi_i, J_k\xi_i) < 0, \ i = 1, \cdots, k, \qquad (\xi_i, J_k\xi_i) > 0, \ i = k+1, \cdots, n.$$

This theorem for the real case is Sylvester's law of inertia ([4], p. 158); but this proof uses a different method. We first state the following:

LEMMA. *If* $\{\xi_1, \cdots, \xi_n\}$ *is linearly independent, and* $(\xi_i, J_k\xi_j) = 0$ *for all* $i \neq j$, *then* $(\xi_i, J_k\xi_i) \neq 0$, $i = 1, \cdots, n$.

The proof is quite simple and will be omitted.

Now to prove the theorem, let $\{\xi_1, \cdots, \xi_n\}$ satisfy the hypothesis above and assume $(\xi_i, J_k\xi_i) < 0$ for $i = 1, \cdots, s$. If $\zeta = \sum_{i=1}^{s} c_i\xi_i$, then

$$(\zeta, J_k\zeta) \leqq 0, \tag{1}$$

since

$$(\zeta, J_k\zeta) = \left(\sum_{i=1}^{s} c_i\xi_i, \sum_{j=1}^{s} c_j J_k\xi_j\right) = \sum_{i=1}^{s} c_i\bar{c}_i(\xi_i, J_k\xi_i),$$

with equality in (1) if and only if every c_i is zero. Now we let P_k be the projection onto the subspace spanned by $\{\alpha_1, \cdots, \alpha_k\}$ *i.e.*, if $\xi = (x_1, \cdots, x_n)$, then $P_k\xi = (x_1, \cdots, x_k, 0, \cdots, 0)$, and let $Q_k = I - P_k$. Consider any set of scalars $c_1, \cdots, c_s$ such that

$$c_1P_k\xi_1 + \cdots + c_sP_k\xi_s = 0.$$

Let $\eta = c_1\xi_1 + \cdots + c_s\xi_s$. Then $\eta = P_k\eta + Q_k\eta = Q_k\eta$, so that

$$(\eta, J_k\eta) \geqq 0. \tag{2}$$

Thus by (1) and (2), $(\eta, J_k\eta) = 0$, and $c_1 = \cdots = c_s = 0$. Thus $\{P_k\xi_1, \cdots, P_k\xi_s\}$ is linearly independent, and $s \leqq k$.

In a similar way, if there are t vectors in the set for which $(\xi_i, J_k\xi_i)$ is positive, then $t \leqq n-k$. But since no $(\xi_i, J_k\xi_i)$ is zero, we must have $s = k$, and $t = n-k$.

3. A model of quasi-Euclidean space. Let R_n be a real n-dimensional Euclidean space with an orthonormal base. We consider the set $\{S\}$, where

$$S = a\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_i x_i + d,$$

with a, d, $a_1, \cdots, a_n$ real numbers, and $(x_1, \cdots, x_n)$ a point of R_n. Then

$$S = \tfrac{1}{2}(a+d)\left(\sum_{i=1}^{n} x_i^2 + 1\right) + \tfrac{1}{2}(a-d)\left(\sum_{i=1}^{n} x_i^2 - 1\right) - 2\sum_{i=1}^{n} a_i x_i.$$

Thus to S corresponds an ordered $(n+2)$-tuple: $[\frac{1}{2}(a+d), \frac{1}{2}(a-d), a_1, \cdots, a_n]$. We may consider for such $(n+2)$-tuples the quasi-inner product of order one. This allows an interesting geometric application. To an element $S \in \{S\}$ for which $a, a_1, \cdots, a_n$ are not all zero, corresponds the n-sphere $s=0$. If the radius of this sphere is r, then $(S, S)_1 = 0$ if and only if $r=0$. Next, if $(S_1, S_1) \neq 0$ and $(S_2, S_2) \neq 0$, then

$$\frac{(S_1, S_2)_1}{(S_1, S_1)_1^{1/2}(S_2, S_2)_1^{1/2}} = \cos\theta,$$

where θ is the angle between the spheres. In addition it is easily verified that if for

$$S_1 = a_1\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_{1i}x_i + d_i,$$

$$S_2 = a_2\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_{2i}x_i + d_2$$

we have $(S_1, S_1) = (S_2, S_2) = 0$ and $a_1 = a_2$, then $(S_1 - S_2, S_1 - S_2)^{\frac{1}{2}}$ is the Euclidean distance between the points $S_1 = 0$ and $S_2 = 0$.

4. Examples of quasi-unitary matrices. Consider the translation $x_i = X_i + h_i$, $i = 1, \cdots, n$ on the space R_n of 3. This induces a linear transformation on $\{S\}$ whose matrix is

$$\begin{pmatrix} 1+\frac{1}{2}\sum h_i^2 & -\frac{1}{2}\sum h_i^2 & -h_1 \cdots -h_n \\ \frac{1}{2}\sum h_i^2 & 1-\frac{1}{2}\sum h_i^2 & -h_1 \cdots -h_n \\ -h_1 & h_1 & 1 \quad 0 \\ \vdots & \vdots & \ddots \\ -h_n & h_n & 0 \quad 1 \end{pmatrix},$$

which is quasi-unitary of order one.

It is also true that, to an element of the conformal group of transformations on R_n, there corresponds a matrix of the form pU_1, where p is a real number and U_1 is quasi-unitary of order one. For example, to the inversion $s(\xi) = (p\xi)/|\xi|^2$,

$\xi \neq 0$, $\xi \in R_n$, there corresponds the transformation on S with matrix

$$\begin{pmatrix} \frac{1}{2}(1+p^2) & \frac{1}{2}(1-p^2) & 0 \cdots 0 \\ -\frac{1}{2}(1-p^2) & -\frac{1}{2}(1+p^2) & 0 \cdots 0 \\ 0 & 0 & p \\ \cdot & \cdot & \cdot \quad 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \quad p \end{pmatrix}.$$

5. Quasi-singular values of a matrix. Let $A=(a_{ij})$ be any n-by-n matrix. Then A^*J_kA and AJ_kA^* are both Hermitian. Let $\lambda_1 \geqq \cdots \geqq \lambda_n$ be the characteristic values of A^*J_kA. Then $\lambda_1, \cdots, \lambda_n$ are called left quasi-singular values of A. The characteristic values $\mu_1 \geqq \cdots \geqq \mu_n$ of AJ_kA^* are called right quasi-singular values of A. In general left and right quasi-singular values are not the same.

If $\{\alpha_1, \cdots, \alpha_n\}$ is the original base in V, then $A\alpha_i=(a_{i1}, \cdots, a_{in})$. Concerning the elements a_{ij} of the matrix A we have:

THEOREM 1.

$$\lambda_n \leqq -\sum_{j=1}^{k} |a_{ij}|^2 + \sum_{j=k+1}^{n} |a_{ij}|^2 \leqq \lambda_1.$$

More generally,

THEOREM 2.

$$\lambda_n + \cdots + \lambda_{n-l+1} \leqq \sum_{p=1}^{l}\left(-\sum_{j=1}^{k} |a_{i_pj}|^2 + \sum_{j=k+1}^{n} |a_{i_pj}|^2\right) \leqq \lambda_1 + \cdots + \lambda_l,$$

for any integers $i_1 < \cdots < i_l \leqq n$.

Similar theorems may be stated and proved for the numbers $\mu_1, \cdots, \mu_n$ and the terms in the columns of the matrix A. The proofs in all cases are like those in [1] and [5].

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A PARITY RELATION PARTITIONS ITS FIELD DISTINCTLY

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1. Introduction. A classical result which pervades all of mathematics states that an equivalence relation partitions its field. In this note, we obtain a stronger statement by observing that a weaker hypothesis implies the same result. The result is well known to several mathematicians, but does not appear to have been published.

A *relation* is a set of ordered couples. An *equivalence relation* is reflexive, symmetric, and transitive. Consider a relation R whose field is S. The following are the conventional definitions of the three properties of an equivalence relation, where x, y, z are any elements of S: (1) R is *reflexive* if xRx; (2) R is *symmetric* if xRy implies yRx; (3) R is *transitive* if xRy and yRz imply xRz.

A comprehensive and clear presentation of several different kinds of reflexivity is contained in Copi [2]. We now call (3) "generally transitive"; this is the usual definition of transitivity as given in such authoritative books as [1] and [7]. But there is another definition of transitivity in the mathematical literature. It appears in the books on graph theory by König [5] and on symbolic logic by Lewis and Langford [6].

R is *distinctly transitive* if whenever x, y, z are distinct elements of S for which xRy and yRz, then xRz. It is clear that every generally transitive relation is distinctly transitive, but not conversely. Of course, R is an equivalence relation if and only if it is reflexive, symmetric, and distinctly transitive. It was shown in [3] that these three properties constitute a "very independent" axiom system. We now state the above-mentioned classical result precisely:

THEOREM E. *A relation R over a field S is an equivalence relation if and only if there is a partition of S into subsets S_i such that for any two elements x, $y \in S$, xRy if and only if x and y are members of the same subset S_i.*

2. Parity relations. We now introduce a type of relation which resembles an equivalence relation in its symmetric and transitive aspects, but differs from it with regard to reflexivity.

An *irreflexive relation* R is one in which no element of S is in the relation to itself. A *parity relation* is irreflexive, symmetric, and distinctly transitive. We shall say that a relation R *partitions its field distinctly* if there exists a partition of S into subsets S_i such that for any two distinct elements x, $y \in S$, xRy if and only if x and y lie in the same subset.

THEOREM. *A parity relation partitions its field distinctly.*

We defer the proof to the next section, where it will be readily demonstrated in terms of some concepts from graph theory.

COROLLARY. *A relation is symmetric and distinctly transitive if and only if it partitions its field distinctly.*

Proof. The "only if" part of the corollary is an immediate consequence of the proof of the theorem. The "if" part is obvious since the relation among elements of a set S of "being in the same subset" is certainly an equivalence relation. Hence, it is necessarily symmetric and distinctly transitive.

Theorem E follows at once from this corollary.

3. Some graphical concepts. We find it convenient to prove the theorem of the preceding section by exploiting known elementary results from the theory of directed graphs, or more briefly "digraphs;" see [5] or [4].

A *digraph* is an irreflexive relation. We call the elements of its field the *points* of the digraph and the ordered pairs in the relation are its *lines*, or *directed lines*. There are three possible ways [4] of defining connectedness for digraphs, called strong, unilateral, and weak connectedness. We require only the latter concept. A digraph is *weakly connected* or, more briefly, *weak*, if for any partition of its set of points into two nonempty subsets, there is a line joining a point of one subset with a point of the other. A (directed) *path* in a digraph D is a collection of n distinct points a_i together with the directed lines $\overrightarrow{a_1a_2}, \overrightarrow{a_2a_3}, \cdots, \overrightarrow{a_{n-1}a_n}$; this is a path *from* a_1 *to* a_n. A *semipath* consists of n distinct points together with exactly one from each of the following pairs of lines: $\overrightarrow{a_1a_2}$ or $\overrightarrow{a_2a_1}$, $\overrightarrow{a_2a_3}$ or $\overrightarrow{a_3a_2}$, $\cdots$, $\overrightarrow{a_{n-1}a_n}$ or $\overrightarrow{a_na_{n-1}}$. Obviously a digraph is weak if and only if any two points are joined by a semipath. A *weak component* of a digraph D is a maximal weakly connected subgraph. Clearly, the weak components of a digraph partition its set S of points, and two points are in the same weak component of D if and only if they are joined by a semipath. We are now ready to prove the theorem.

Proof of theorem. Let R be a parity relation whose field is S. We shall also use the letter R for the digraph of this relation. Let $S_1, S_2, \cdots$ be the subsets of points in the weak components of R. Then these subsets S_i constitute a partition of S. Let x and y be any two distinct elements of S. If xRy, then the digraph R contains the directed line $\overrightarrow{xy}$. Thus x and y are certainly in the same weak component.

Conversely, if x and y are distinct points in the same weak component S_i then there exists a semipath joining them. By the symmetry of relation R, we find that there also exists in this weak component a directed path from x to y. Applying the hypothesis of distinct transitivity, it follows that the line $\overrightarrow{xy}$ lies in this weak component; hence, xRy.

The consequence of this theorem is that the distinct partitioning of a set is independent of reflexivity. Thus, if we have a relation R which is neither reflexive nor irreflexive, there still results a distinct partitioning of its field provided R is symmetric and distinctly transitive.

An example of a parity relation among people is provided by: x is a sibling of y. For by definition this relation is symmetric and distinctly transitive, but irreflexive.

A *graph* is an irreflexive symmetric relation. A *complete graph* is one in which any two distinct points lie in the relation to each other. The graphical interpretation of a parity relation is the following: *R is a parity relation if and only if each of its weak components is a complete graph.*

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AN OPTIMUM SHAPE FOR FAIRING THE EDGE OF AN ELECTRODE

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1. Introduction. Application of conformal mapping to electromagnetic theory. Boundary-value problems involving the two-dimensional Laplace equations, which arise in many physical problems, can often be solved with the aid of conformal mapping. For suppose that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

over some domain. Then we can define the complementary function $\psi(x, y)$ such that $\chi=\phi+i\psi$ is an analytic function of $z=x+iy$. Suppose also that $z=f(w)$, $w=u+iv$, is an analytic function. Then $\chi[f(w)]=\phi[f(w)]+i\psi[f(w)]$ is also analytic, and hence

$$\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}=0.$$

If the function $z=f(w)$ is a 1-1 mapping over some domain of z and w, its inverse transforms solutions of Laplace's equation in the $xy-$plane into solutions in the $uv-$plane, with equipotential lines being transformed into equipotential lines and so on. This technique is useful because it enables us to transform problems involving a complicated boundary into problems involving a simple boundary such as a straight line or a circle, for which special solutions of the Laplace equation are available.

Consider, for example, a region bounded by two straight lines which join at an angle α (Fig. 1)

$$w = u + iv = re^{i\theta}, \qquad r \geqq 0, \qquad 0 \leqq \theta \leqq \alpha \leqq 2\pi,$$

and let us seek a solution to the electrostatic potential problem $\phi(u, v)$ which is zero on the boundary.* The mapping function $z = w^{\pi/\alpha}$ maps the region in the w-plane into

$$z = (re^{i\theta})^{\pi/\alpha} = r^{\pi/\alpha}e^{i\pi\theta/\alpha} = \rho e^{i\psi}, \qquad \rho \geqq 0, \qquad 0 \leqq \psi \leqq \pi,$$

i.e., into the upper half plane. An example of a simple potential function which vanishes on the boundary is $\phi(x, y) = y$ which is the potential of a uniformly charged plane at $y = 0$.

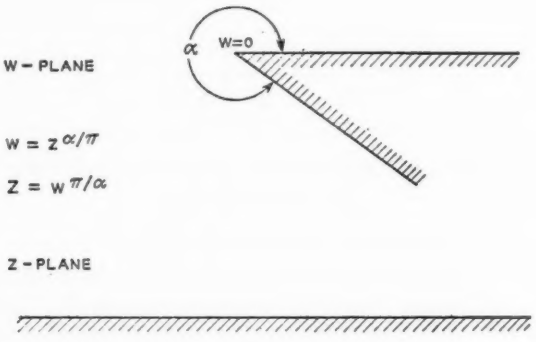

FIG. 1

In this case,

$$\phi(u, v) = y(u, v) = \text{Im}\,(u + iv)^{\pi/\alpha} = (u^2 + v^2)^{\pi/2\alpha} \sin\left(\frac{\pi}{\alpha} \tan^{-1}\frac{v}{u}\right)$$

is a solution meeting the boundary conditions.

It is often convenient to retain the whole analytic function $\chi(w) = \phi(w) + i\psi(w)$, instead of only the real part $\phi(w)$. For example, if ϕ is an electrostatic potential and $\mathbf{E}$ the corresponding electric field vector, then $\mathbf{E} = -\text{grad}\,\phi$. But

$$\frac{d\chi}{dw} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial w} + i\left[\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial w}\right].$$

Eliminating ψ by the Cauchy-Riemann equations,

* We are really imagining a three-dimensional problem, where all parameters are independent of the third orthogonal space-variable; the uv-plane is a cross-section.

$$\frac{\partial \phi}{\partial u} = \frac{\partial \psi}{\partial v}, \qquad \frac{\partial \phi}{\partial v} = -\frac{\partial \psi}{\partial u},$$

we find

$$\frac{d\chi}{dw} = \frac{\partial \phi}{\partial u} - i\frac{\partial \phi}{\partial v} = \overline{\text{grad } \phi},$$

where the bar denotes the complex conjugate. Hence $\mathbf{E} = \overline{d\chi/dw}$. If the boundary is an equipotential, the surface charge density is

$$\rho = |\mathbf{D}| = \epsilon |\mathbf{E}| = \epsilon \left| d\chi/dw \right|,$$

where $\mathbf{D}$ is the electric displacement vector and ϵ is the dielectric constant of the medium surrounding the conductor. In the present case:

$$\chi = -ix + y = -iz = -iw^{\pi/\alpha},$$

$$d\chi/dw = (-\pi i/\alpha)w^{\pi/\alpha - 1},$$

$$\rho = \epsilon \left| d\chi/dw \right| = (\pi\epsilon/\alpha)(u^2 + v^2)^{\pi/(2\alpha) - 1/2}.$$

(Although $\chi = -iz$ in the present illustrative example, this is not true in general.)

It is clear from this example that if $\alpha > \pi$, *i.e.*, if the corner is convex, the charge density is unbounded in the neighborhood of the corner. In a practical case this leads to such phenomena as corona discharge.

In the case where the figure represents the cross-section of a uniform cylindrical (not necessarily circular) electrical transmission line with perfectly conducting boundaries, we can represent a solution of Maxwell's equations by

$$\mathbf{D}/\epsilon = \mathbf{E} = -\text{grad } \phi e^{i\omega t - \gamma\zeta}, \mu\mathbf{H} = \mathbf{B} = \sqrt{(\mu\epsilon)}\ \text{grad } \psi e^{i\omega t - \gamma\zeta},$$

where

$\mathbf{D}$ = electric displacement vector,
$\mathbf{E}$ = electric intensity vector,
$\mathbf{H}$ = magnetic intensity vector,
$\mathbf{B}$ = magnetic flux-density vector,
μ = permeability of the medium,
ϵ = permittivity of the medium (*i.e.*, absolute dielectric constant),
ω = frequency in radians per second,
$\gamma^2 = -\omega^2\mu\epsilon$,
ζ = a coordinate perpendicular to the section, *i.e.*, along the axis of the transmission line, and
t = time.

From the above we can deduce that the surface current density at the bounding surfaces is

$$|\mathbf{j}| = \sqrt{(\epsilon/\mu)}\,|\text{grad } \phi| = (\pi/\alpha)\sqrt{(\epsilon/\mu)}(u^2 + v^2)^{\pi/(2\alpha) - 1/2}.$$

If we imagine a small but finite resistivity in the bounding conductors, the resistance losses in the conductor are large in the neighborhood of the corner.

2. Statement of the problem. To remove the singularity at the edge of a conductor. In the examples cited above, the charge density and current density have an infinite singularity at the sharp corner. For practical reasons, this is often undesirable. As long as the corner is sharp, the singularity must remain. The only means to overcome the effects of the singularity is to use a smooth curve of some sort to go around the bend. But what curve?

A reasonable solution to seek is one in which the value of $|d\chi/dw|$ attains but does not exceed a fixed maximum. Intuitively we realize that a convex surface builds up charge (or current) density in some way related inversely to its radius of curvature. Hence to go around the corner as economically as possible, we should look for a curve for which $|d\chi/dw|$ is constant going around the curve.

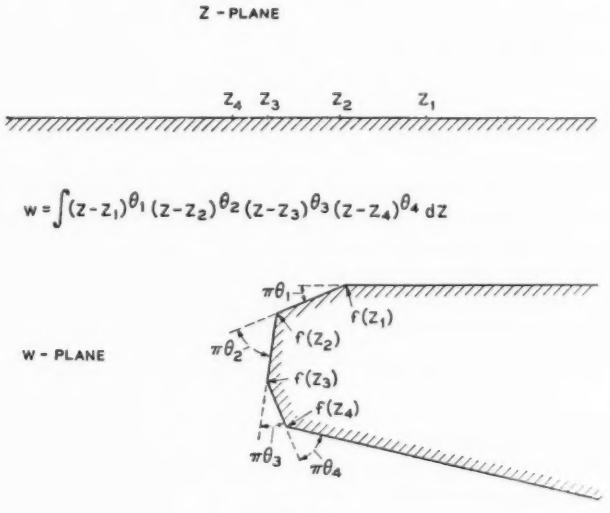

FIG. 2
THE SCHWARTZ-CHRISTOFFEL TRANSFORMATION

3. Solution of the mapping problem. At first sight the problem does not seem tractable. We could, however, consider a curve made up of a polygonal arc, and gradually smooth the polygon by increasing the number of sides. Using the Schwartz-Christoffel transformation

$$w = \int (z - z_1)^{\theta_1} \cdots (z - z_n)^{\theta_n} dz, \qquad z_j \text{ real},$$

we transform the upper half plane into a polygonal region with n exterior angles

$\pi\theta_1, \pi\theta_2, \cdots, \pi\theta_n$ (see Fig. 2). For simplicity, let all the angles be equal. Recalling that the angle between the semi-infinite straight sides is α, we find that $\pi\theta_1 + \cdots \pi\theta_n = n\pi\theta = \alpha - \pi$, $\theta = (\alpha - \pi)/n\pi$. Letting

$$P_n(z) = \prod_{j=1}^{n} (z - z_j),$$

we find

$$w = \int [P_n(z)]^{(\alpha/\pi - 1)/n} dz.$$

If as before $\chi(z) = z$, then

$$\frac{d\chi}{dw} = \frac{dz}{dw} = [P_n(z)]^{(1-\alpha/\pi)/n}.$$

The region on the real z-axis corresponding to the multiple corner is the region where the (real) zeros of $P_n(z)$ occur. Inasmuch as the exponent of P_n in the above equation is negative, the value of $|d\chi/dw|$ is unbounded near each of these zeros. However, we can still pursue the idea of making $|d\chi/dw|$ as nearly constant as possible, in some sense, and see where it leads.

In some intuitive sense the Tschebyscheff* polynomials are most nearly constant over an interval; for in an interval $(-1, 1)$ containing their zeros, the maximum absolute value of each is unity, and this is attained once between each pair of zeros. No other polynomials of the same degree and same mean-square magnitude have their extreme excursion confined to such narrow limits. Further amplification of the line of reasoning is unnecessary, for the result amply justifies the questionable logic. Following this heuristic notion, let

$$P_n(z) = T_n(z) = \tfrac{1}{2}[\{z + \sqrt{(z^2 + 1)}\}^n + \{z - \sqrt{(z^2 - 1)}\}^n].$$

The representation chosen is particularly convenient, for

$$\lim_{n \to \infty} [T_n(z)]^{1/n} = z + \sqrt{(z^2 - 1)}, \qquad \text{Im } z > 0.$$

We can therefore look more carefully at the mapping function

$$w = \int^{z} [z + \sqrt{(z^2 - 1)}]^a dz \qquad (a = \alpha/\pi - 1)$$

$$= \begin{cases} \tfrac{1}{2}[z^2 - z\sqrt{(z^2 - 1)} + \log\{z - \sqrt{(z^2 - 1)}\}] & (a = -1), \\ \dfrac{[z + \sqrt{(z^2 - 1)}]^a [z - a\sqrt{(z^2 - 1)}]}{1 - a^2} & (-1 < a < 1), \\ \tfrac{1}{2}[z^2 + z\sqrt{(z^2 - 1)} - \log\{z + \sqrt{(z^2 - 1)}\}] & (a = +1). \end{cases}$$

* I am indebted to Bruce P. Bogert for suggesting the use of Tschebyscheff polynomials.

We choose the branch of $\sqrt{(z^2-1)}$ which is positive real for real $z>1$, negative real for real $z<-1$, and which has positive imaginary part for z in the upper half plane. It is clear from direct calculation that for $-1<z<+1$,

$$|\, d\chi/dw \,| \;=\; |\, dw/dz \,|^{-1} = \left[z^2 + \left\{\sqrt{(1-z^2)}\right\}^2\right]^{-\alpha/2} = 1.$$

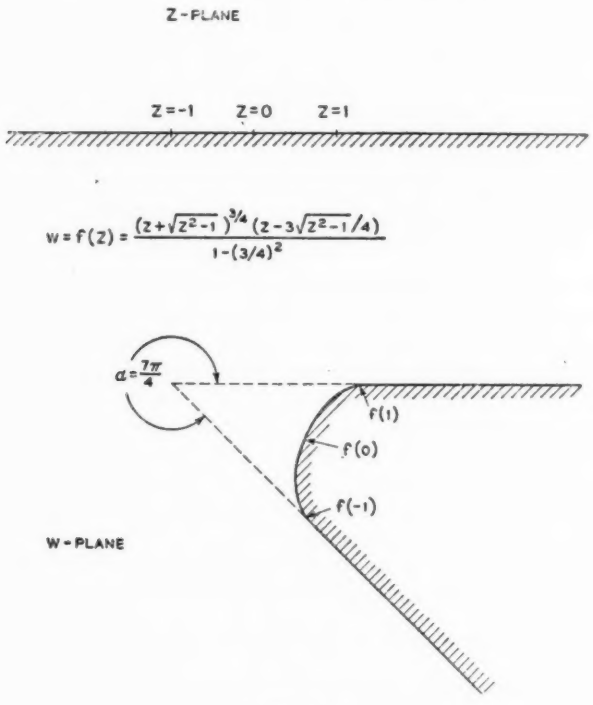

FIG. 3
AN EXAMPLE OF MAPPING FUNCTION FOR $\alpha=7\pi/4$

In fact, we can deduce that for the general case $-1<a<1$, *i.e.*, $0<\alpha<2\pi$,

$1 < z < \infty$ (real z) maps into $w = \rho e^{i0}$, $(1-a^2)^{-1} < \rho < \infty$;

$-\infty < z < -1$ (real z) maps into $w = \rho e^{i\alpha}$, $\infty > \rho > (1-a^2)^{-1}$;

$-1 < z < 1$ (real z) maps into a hypocycloid

$$w = \frac{-(1+a)\cos(1-a)\theta + (-1+a)\cos(1+a)\theta}{2(a^2-1)} + i\,\frac{(a-1)\sin(1+a)\theta + (a+1)\sin(1-a)\theta}{2(a^2-1)},$$

where $z=\cos\theta$. For $a=\pm 1$,

$1 < z < \infty$ (z real) maps into $w = \rho e^{i0}$, $\quad \frac{1}{2} < \rho < \infty$;

$-\infty < z < -1$ (z real) maps into $w = e^{i0} \mp \frac{1}{2}\pi i$, $\quad \infty > \rho > \frac{1}{2}$;

$-1 < z < 1$ (z real) maps into a cycloid $w = \frac{1}{4}[1 + \cos 2\theta \mp i(2\theta - \sin 2\theta)]$.

An example of the map, for $\alpha=7\pi/4$, $a=3/4$, is shown in Figure 3. In geometric terms, the result is always two straight lines emanating with no change in direction from the extremities of a single leaf of a cycloid or a hypocycloid.

4. Uniqueness of the mapping function. It is interesting to note that, except for a bilinear transformation in the z-plane which maps the upper half plane into itself, and translations and rotations in the w-plane, the mapping found above is unique. Inasmuch as the proof is less interesting than the fact, it will only be sketched here.

Let $w=f(z)$ be a mapping which maps the upper half plane into a region bounded by portions of the lines

$$w = re^{0},\ r > r_1, \qquad w = re^{i\alpha},\ r > r_2$$

and some arc joining them, such that

$$f(\infty) = \infty, \qquad f(1) = r_1, \qquad f(-1) = r_2 e^{i\alpha},$$

and

$$|df/dz| = k \qquad \text{for } -1 < z < 1.$$

Let

$$g(z) = \int^{z} \{z + \sqrt{(z^2 - 1)}\}^{a} dz$$

as above. Because straight lines are analytic arcs, f and g are both regular for real z, $1<|z|<\infty$, as well as for z with positive real part. Consideration of $g^{-1}[f(z)]$ in the neighborhood of $z=\infty$ shows that it must have a simple pole with real residue. Hence $h(z)=f'(z)/g'(z)$ is analytic in the upper half plane, at ∞, and on the real axis except for $-1<z<1$. It can be extended by analytic continuation, thus: $h(z),=\overline{h(\bar{z})}$ to include the whole plane save a cut on the real axis $-1<z<1$. Furthermore, since f is a mapping function, $f'(z)$ is never zero, and hence neither is $h(z)$. Hence by the maximum modulus theorem, $|h(z)|$ attains its maximum and minimum values on the boundary $-1<z<1$. But over this range

$$|h(z)| = |f'(z)/g'(z)| = k/1.$$

Hence $h(z)$ is a constant, and $f(z)$ is a linear function of $g(z)$, the function already determined.

For $a=\pm 1$, the above argument needs a trifling modification. In this case, the half plane is mapped by f or by g on to the inside or outside of a semi-infinite

parallel-sided strip, and for the above argument to work the widths of the strips must be the same.

5. Other properties of the mapping function. Continuation across the boundary. The boundary described above is by no means an ultimate boundary. It is in fact made up of three analytic arcs joined at three point singularities (including one at ∞). The mapping function can be extended across any one of the three arcs.

The extensions across the straight lines are not very interesting, for they lead to simple geometric reflections. However, the continuation across the cycloid does lead to an interesting result. Suppose, for example, we make a cut in the z-plane from $+1$ to $+\infty$ and from -1 to $-\infty$ along the real axis, and extend the mapping across $(-1, +1)$. Formally the integrations giving w as a function of z are still correct, provided we choose the appropriate branch of the square root. If we cross the axis and return to a real value to z, $z>1$, from below, clearly we take the other branch. But

$$z - \sqrt{(z^2 - 1)} = \{z + \sqrt{(z^2 - 1)}\}^{-1}.$$

Hence, the effect is to change a into $(-a)$ in the above representation. Following this argument through shows that the whole z-plane with the above two slits is mapped into the whole w-plane with two slits making an angle α (the slits being the straight portions of the boundary already found). Such a map will have applications in potential problems involving two sheets nearly joined at the edge, or fluid-flow problems from some orifices.

6. A general property of polynomial expansions. The above analysis is related in an interesting way to the theory of general polynomial expansions. Suppose we are given a schlicht, bounded, simply-connected domain B and a set of polynomials $P_n(z)$ such that every function $f(z)$ regular on B can be represented in an expansion of the form $f(z) = \sum b_n P_n(z)$ convergent at every point of B. Then, with further restrictions about the absolute scale of $P_n(z)$, the nature of the domain B, how and whether the expansion converges on the boundary of B, and so forth, Walsh and others [1], [2] have shown that in some sense

$$\lim_{n\to\infty} [P_n(z)]^{1/n} = g(z),$$

where $g(z)$ is the exterior mapping function which maps the exterior of B into the exterior of the unit circle. If B is the line segment from $(-1, 0)$ to $(+1, 0)$, then

$$\lim_{n\to\infty} [P_n(z)]^{1/n} = g(z) = z + \sqrt{(z^2 - 1)}. \tag{1}$$

For this case, Szegö [3] has proved the following:

Let $\{P_n(x)\}$, $n=1, 2, \cdots$, *be a set of polynomials of degree* n *respectively, orthonormal in* $(-1, 1)$ *with a weighting factor* $w(x)$ *such that*

$$\int_{-\pi}^{\pi} |\log \{\sin\theta\, w(\cos\theta)\}|\, d\theta$$

exists. Then for all x not in the real interval $(-1, 1)$,

$$P_n(x) \simeq \frac{[x + \sqrt{(x^2-1)}]^n}{\sqrt{(2\pi)}\, D[x - \sqrt{(x^2-1)}]},$$

where

$$D(u) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log[\sin\theta\, w(\cos\theta)]\, \frac{1+ue^{i\theta}}{1-ue^{i\theta}}\, d\theta\right\}, \tag{2}$$

and furthermore $D(u) \neq 0$ *for* $|u| < 1$, *and* $D(0)$ *is real and greater than zero. The branch of the square root is that which approaches x as* $|x| \to \infty$.

Obviously this implies (1) for the case at hand, and actually somewhat more. Hence in the heuristic argument aired earlier, the polynomials of Legendre, for example, might have been used instead of those of Tchebyscheff, with the same final result. The reader may satisfy himself, for example, by looking in the cited work of Szegö, that (1) is not trivial for Legendre or many other polynomial sets. In spite of the fact that many different choices of polynomials would suffice to find the desired conformal map, the heuristic choice of Tchebyscheff polynomials is upheld: they are the best choice because they make $D(u)$ [eq. (2)] constant.

7. Conclusions and applications. In a certain sense, the mapping described above answers the question, what is the best way to shape the corner of a conductor to relieve electrical stresses at sharp edges? It can be applied directly to the shaping of high-voltage bus bars and connected straps, to the shaping of supports located near high-voltage conductors and to the shaping of the edges of a conductor in a strip transmission line. It may also be applied to the design of contours of channels carrying fluid flow where reduction of turbulence may be desired, such as a bend in a canal or the joint between a conical and a cylindrical portion of a projectile or missile. In cases where the ideal conditions of the mathematical problem are not completely valid, the choice of a hypocycloidal curve to fair a corner may still give a better result than a circular arc, or may lead more easily to a suitable empirical solution.

I am indebted to H. O. Pollak, B. P. Bogert, and A. Pal for their counsel and assistance.

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THE ERDÖS INEQUALITY AND OTHER INEQUALITIES FOR A TRIANGLE

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1. It is well known that, if P is a point in a triangle ABC whose distances are x, y, z from the vertices and p, q, r from the sides, a remarkable inequality connects the six numbers x, y, z, p, q, r:

$$x + y + z \geqq 2(p + q + r) \tag{1}$$

with equality if and only if the triangle is equilateral and P is its center. This inequality was found by Erdös* and proved by Mordell and Barrow [1].

I have obtained many other inequalities connecting these distances, *e.g.*,

$$px + qy + rz \geqq 2(qr + rp + pq), \tag{2}$$

$$yz + zx + xy \geqq 4(qr + rp + pq), \tag{3}$$

$$xyz \geqq 8pqr, \tag{4}$$

$$2^{-1}(p^{-1} + q^{-1} + r^{-1}) \geqq x^{-1} + y^{-1} + z^{-1}. \tag{5}$$

For an internal point P equality can only occur when the triangle is equilateral and P is its center. There is also equality at the vertices but not on the sides.

It will be seen that (1), (2), (5) are equivalent in the sense that each can be obtained from the others by using well-known geometrical transformations. I shall prove (2) directly and then use the following transformations: (i) isogonal conjugates, (ii) inversion, (iii) reciprocation. For simplicity I shall take P as the center for inversion and for reciprocation.

Before beginning the proofs I note that in *Mathematical Reviews*, vol. 21, 1960, p. 162 it is stated that A. Florian has proved (in my notation) the inequality

$$x^k + y^k + z^k \geqq 2^k(p^k + q^k + r^k) \tag{6}$$

for $|k| \leqq 1$ with equality as in (1).

That (6) holds for $0 < k < 1$ is certainly true (as I shall prove): for $k = 0$, (6) is an identity: for $-1 \leqq k < 0$, (6) is false as my inequality (5) indicates. In fact for $-1 \leqq k < 0$ the signs of inequality in (6) must be reversed. So too for the other inequality of Florian:

$$x^k + y^k + z^k > 2(p^k + q^k + r^k) \qquad (|k| > 1). \tag{7}$$

It is true for $k > 1$: for $k < -1$ the sign of inequality must be reversed.

2. Proof of (2) and (4). We begin with the elementary inequality

* Erdös first proposed his inequality as a problem in this MONTHLY in 1935. In 1937 two proofs were published, one by Mordell and one by Barrow. Barrow indeed proves a stronger inequality which I propose, in another note, to extend in a remarkably simple way.

$$ax \geqq bq + cr, \tag{8}$$

equality holding if and only if P is on the altitude from A. If h_1 is this altitude, then

$$a(x + p) \geqq ah_1 = 2\Delta = ap + bq + cr, \tag{9}$$

whence (8) follows at once.

From (8), $px \geqq (b/a)pq + (c/a)pr$ with equality if and only if P is on the altitude from A or if P is on BC. Hence

$$\sum px \geqq \sum \left(\frac{b}{c} + \frac{c}{b}\right) qr \tag{10}$$

with equality if and only if P is a vertex or if (the triangle being acute-angled) P is the orthocenter.

But $b/c + c/b \geqq 2$ by the theorem of the means so that (10) yields

$$px + qy + rz \geqq 2(qr + rp + pq) \tag{11}$$

for any point P in a triangle ABC. Equality holds when P is a vertex or when the triangle is equilateral and P is its center. This proves (2).

To prove (4) apply the inequality of the means directly to (8). Then $ax \geqq 2(bcqr)^{\frac{1}{2}}$ and two similar inequalities. Multiplication yields

$$xyz \geqq 8pqr, \tag{12}$$

with equality as in (2). This proves (4).

3. I now prove the inequality

$$(px)^k + (qy)^k + (rz)^k \geqq 2^k((qr)^k + (rp)^k + (pq)^k) \tag{13}$$

for $0 < k < 1$, equality occurring as for (11). To prove (13), I apply to (8) the well-known elementary inequality

$$\left(\frac{u + v}{2}\right)^k \geqq \frac{u^k + v^k}{2} \qquad (u > 0, v > 0, 0 < k < 1), \tag{14}$$

equality occurring if and only if $u = v$. Thus (8) yields

$$(px)^k \geqq 2^{k-1}\left\{(b/a)^k(pq)^k + (c/a)^k(pr)^k\right\}, \tag{15}$$

whence

$$\sum (px)^k \geqq 2^{k-1} \sum \left\{\left(\frac{b}{c}\right)^k + \left(\frac{c}{b}\right)^k\right\} (qr)^k, \tag{16}$$

with equality as for (10). From (16) by the argument which led from (10) to (11) we obtain (13).

4. Isogonal conjugates. If P and P' are such that AP, AP' are equally in-

clined to the bisectors of A; BP, BP' equi-inclined to the bisectors of B; CP, CP' to those of the angle C, then P and P' are isogonal conjugates. If accented letters refer to P', then

$$pp' = qq' = rr', \qquad p'\sum aqr = 2\Delta qr, \qquad p\sum aq'r' = 2\Delta q'r'. \tag{17}$$

$$x'\sum aqr = 2\Delta px, \qquad x\sum aq'r' = 2\Delta p'x'. \tag{18}$$

The proof is simple. In trilinear coordinates the isogonal property yields

$$\frac{p'}{1/p} = \frac{q'}{1/q} = \frac{r'}{1/r} = \frac{2\Delta}{\sum a/p} = \frac{2\Delta pqr}{\sum aqr}$$

whence (17) follows. As for (18) it is enough to note that

$$x \sin A = (q^2 + r^2 + 2qr\cos A)^{1/2},$$

$$x' \sin A = (q'^2 + r'^2 + 2q'r'\cos A)^{1/2} = (r^2 + q^2 + 2rq\cos A)^{1/2} 2\Delta pqr/qr\sum aqr,$$

so that $x'\sum aqr = 2\Delta px$, which is the first equation in (18).

It will be observed that if P is an *internal* point of the triangle so also is P' and vice-versa. Hence *any homogeneous inequality (or equality) connecting x, y, z, p, q, r for any internal point P leads to another homogeneous inequality (or equality) merely by the substitution*

$$S: \quad (x, y, z; p, q, r) \to (px, qy, rz; qr, rp, pq). \tag{19}$$

Thus (11) yields $\sum qrpx \geqq 2\sum rppq$ or $\sum x \geqq 2\sum p$, the Erdös inequality: and conversely the Erdös inequality yields (11). But (12) is clearly self-dual for this transformation.

If we apply S to (13) it yields the inequality

$$\sum x^k \geqq 2^k \sum p^k \qquad (0 < k < 1) \tag{20}$$

given by Florian. (My knowledge of Florian's result derives from *Mathematical Reviews* cited above: his paper is not accessible to me.)

5. Inversion. I consider next inversion and, for simplicity, inversion with respect to the internal point P itself. Here we obtain a new triad A', B', C' corresponding to A, B, C: P is internal to the triangle $A'B'C'$. Using accents to denote the distances of P relating to the triangle $A'B'C'$ it is clear that

$$\begin{aligned} xx' &= K^2, \qquad yy' = K^2, \qquad zz' = K^2 \\ p'yz &= K^2 p, \qquad q'zx = K^2 q, \qquad r'xy = K^2 r \end{aligned} \tag{21}$$

where K is the radius of inversion.

(For $p' \cdot B'C' = 2\Delta B'PC' = PB' \cdot PC' \cdot B'C'/$(diameter of circle $B'PC'$): diameter $= K^2/p$.)

It follows therefore that *inequalities (or equalities) which involve x, y, z, p, q, r homogeneously remain true under the substitution denoted by*

$$V: \quad (x, y, z; p, q, r) \to (yz, zx, xy; px, qy, rz). \tag{22}$$

Apply V to the Erdös inequality: we obtain the new inequality

$$yz + zx + xy \geqq 2(px + qy + rz) \tag{23}$$

which, combined with (11), yields another inequality

$$yz + zx + xy \geqq 4(qr + rp + pq). \tag{24}$$

Thus (1), (24), (12) show that the three elementary symmetric functions of x, y, z are at least equal to the corresponding elementary symmetric functions of $2p, 2q, 2r$.

V applied to (11) gives rise to a new inequality

$$xyz(p + q + r) \geqq 2\sum qryz \tag{25}$$

or $2^{-1}\sum q^{-1}r^{-1} \geqq \sum p^{-1}x^{-1}$.

Apply S to (25): we obtain (after obvious reduction)

$$p^{-1} + q^{-1} + r^{-1} \geqq 2(x^{-1} + y^{-1} + z^{-1}) \tag{26}$$

which is enough to show that (6), so far as $k = -1$ is concerned, must have the sign of inequality reversed.

6. Reciprocation. I consider next reciprocation and again for simplicity reciprocation with respect to P. We obtain a triangle $A''B''C''$; P is still an internal point. The distances of P from the sides of $A''B''C''$ are inversely proportional to its distances from the vertices of ABC: its distances from A'', B'', C'' are inversely proportional to its distances from the sides of ABC.

It follows that *inequalities (or equalities) involving x, y, z, p, q, r homogeneously remain valid under the substitution*

$$R: \quad (x, y, z, p, q, r) \to (p^{-1}, q^{-1}, r^{-1}; x^{-1}, y^{-1}, z^{-1}). \tag{27}$$

As an example apply R to the Erdös inequality: we obtain (26) at once.

As another example apply R to (20): we obtain $\sum p^{-k} \geqq 2^k \sum x^{-k}$ or

$$2^{-k}\sum p^{-k} \geqq \sum x^{-k} \qquad (0 < k < 1) \tag{28}$$

which shows that Florian's inequality (6) needs reversal of sign of inequality when the k of (6) is negative.

7. Corresponding to any one inequality we obtain a set of six inequalities (which may coincide) by applying these transformations. The set which arises from the Erdös inequality is the following:

$$\sum x \geqq 2\sum p, \qquad \sum px \geqq 2\sum qr,$$
$$\sum yz \geqq 2\sum px, \qquad \sum 1/p \geqq 2\sum 1/x,$$
$$\sum 1/qr \geqq 2\sum 1/px, \qquad \sum 1/px \geqq 2\sum 1/yz.$$

On the other hand the inequality $xyz \geqq 8pqr$ gives rise to no new inequality, while, in contrast, the inequality

$$xyz \geqq (q + r)(r + p)(p + q) \tag{29}$$

(not proved here) gives rise to a set of three inequalities.

8. Other inequalities can be obtained by applying the transformations to such inequalities as $ax \geqq bq + cr$ (if we make the necessary changes for a, b, c) or by inverting or reciprocating with respect to points other than P.

Two invariants of the group of transformations may be noted:

$$\text{(i) } J = pqr/xyz, \qquad \text{(ii) } H = a/p + b/q + c/r. \tag{30}$$

9. I state also a surprising theorem I have found about a triangle G and a point P in its plane (not on its sides).

Let G_1 be the triangle formed by the feet of the perpendiculars from P on the sides of the triangle G. Form G_2 from G_1 and P in the same way. Likewise construct G_3 from G_2 and P. Then the triangle G_3 so constructed is necessarily similar to the original triangle G.

This theorem, which arises incidentally from the group of transformations considered, extends also to plane polygons: the nth polygon constructed being similar to the original polygon.

10. The inequality (2) extends at once to n dimensions: by the same proof as in Section 2 I find that

$$\sum_{i=1}^{n+1} p_i x_i \geqq 2 \sum_{1 \leq i < j \leq n+1} p_i p_j; \tag{31}$$

equality occurs at a vertex or, if the "tetrahedron" is regular, at the center: there is no other case of equality.

Corresponding to (4) is the inequality

$$x_1 x_2 \cdots x_{n+1} \geqq n^{n+1} p_1 p_2 \cdots p_{n+1}. \tag{32}$$

Other inequalities, not very easy to write down, arise by inversion and reciprocation.

11. I am indebted to Dr. Diananda and Dr. Guha for pointing out to me that (3), (4), (5) and also reciprocation are to be found in Fejes-Tóth [2]. To Professor Pedoe I express my thanks for helpful criticism.

I conclude by stating without proof an inequality much stronger than (3):

$$\sum yz \geqq \sum (p + q)(p + r). \tag{33}$$

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A DEVELOPMENT OF A SERIES STUDIED BY H. W. GOULD

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1. **Introduction.** The purpose of this note is to draw attention to certain properties of the series

$$\beta_t \equiv \beta_{t,p}(x) \equiv 1 + \binom{t+p}{1}x + \cdots + \binom{t+rp}{r}x^r + \cdots, \tag{1}$$

and of allied series, and to develop some of their consequences. The analysis will lead to an explicit solution of all the $m+n$ roots of $x(1+\theta)^{m+n}-\theta^n=0$. The series $\int_0^x \beta_p dx$, which will be denoted by ϕ, or $\phi_p(x)$, plays an important part in the theory. Thus

$$\phi \equiv \phi_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{jp-j+1}\binom{jp}{j}x^j = \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp}{j-1}x^j \tag{2}$$

$$\phi' = \beta_p. \tag{3}$$

2. **A property of β_t and some consequences.** The series (1) and (2) are of the type studied by H. W. Gould ([3], [4]), who has shown ([3], eqs. (7), (9)) that in our notation, and for all real values of t:

$$\beta_t = \frac{(1+\phi)^{t+1}}{1+\phi-p\phi}, \tag{4}$$

$$x(1+\phi)^p = \phi, \tag{5}$$

$$(1+\phi)^t = \sum_{j=0}^{\infty}\binom{t+pj}{j}\frac{t}{t+pj}x^j, \tag{6}$$

where $|x| < |(p-1)^{p-1}/p^p|$.

Writing $t=0$ in (4) we have

$$\beta_0 = \frac{1+\phi}{1+\phi-p\phi} \tag{7}$$

and hence,

$$\beta_t = (1+\phi)^t\beta_0. \tag{8}$$

Thus β_t is a geometric progression, for successive values of t, the common ratio $1+\phi$ being closely allied to a particular term in the progression, *viz.*, β_p, by (3). From (8), with $t=p-1$, we have

$$\beta_{p-1} = \frac{\beta_p}{1+\phi} = \frac{\phi'}{1+\phi}.$$

Integrating, we obtain

$$F \equiv F_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp-1}{j-1}x^j = \log(1+\phi)$$

since both F and ϕ vanish when $x=0$; *i.e.*,

$$e^F = 1+\phi. \tag{9}$$

Similarly from (8) with $t=p-2$, we have

$$\beta_{p-2} = \frac{\beta_{p-1}}{1+\phi} = \frac{\phi'}{(1+\phi)^2}.$$

Integrating, we obtain

$$\psi \equiv \psi_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp-2}{j-1}x^j = \frac{-1}{1+\phi}+1$$

since both ψ and ϕ vanish when $x=0$; hence

$$(1+\phi)(1-\psi) = 1, \tag{10}$$

$$e^{-F} = 1-\psi, \tag{11}$$

and from (5) and (10),

$$x = \psi(1-\psi)^{p-1}. \tag{12}$$

Also from (7) and (10),

$$\beta_0 = \frac{1+\phi}{1-(p-1)\phi} = \frac{1}{1-p\psi}. \tag{13}$$

These relations were deduced by Bizley [1] as a special case of a more general investigation, by combinatorial arguments based upon the geometrical properties of lattice paths.

3. **Generating function of β_t ($t \geqq 0$).** Define

$$\begin{aligned}
[\beta]_t &= \beta_t && (t \geqq 0),\\
&= \beta_t - 1 && (-p+1 \leqq t < 0),\\
&= \beta_t - 1 - \binom{t+p}{1}x && (-2p+2 \leqq t < -p+1),\\
&= \beta_t - 1 - \binom{t+p}{1}x - \binom{t+2p}{2}x^2 && (-3p+3 \leqq t < -2p+2),
\end{aligned}$$

and so on. Thus $[\beta]_t$ is that part of β_t which starts with the first term $\binom{t+rp}{r}x^r$, where $t+rp \geqq r$.

Consider $s+x/s^{p-1}$ for positive x,s. This takes the value ∞ when $s=0$, drops

to a minimum when $s^p = (p-1)x$ and rises again to ∞ when $s = \infty$. If $x<(p-1)^{p-1}/p^p$ it will be found that the minimum value of $(s+x/s^{p-1})$ falls below unity, and thus there are then two and only two real positive roots of the equation $s+x/s^{p-1}=1$ or $s^p-s^{p-1}+x=0$. Denote them by α_1 and $\alpha_2(\alpha_2<\alpha_1)$. (From (12), one of these roots is $(1-\psi)$ and it will be seen later that, in fact, $1-\psi=\alpha_1$).

Now it is easy to verify that, provided $s+x/s^{p-1}<1$,

$$E \equiv 1/\{1-(s+x/s^{p-1})\} = \sum_{t=-\infty}^{\infty} [\beta]_t s^t. \tag{14}$$

Alternatively $E=-s^{p-1}/(s^p-s^{p-1}+x)=\sum_{r=1}^{p} A_r/(1-\gamma_r s)$, where A_r is a constant and $\gamma_1, \cdots, \gamma_p$ are the roots of $x\theta^p-\theta+1=0$ and where, to fix ideas, we will take $\gamma_1=1/\alpha_1$, $\gamma_2=1/\alpha_2$ (whence $\gamma_1<\gamma_2$).

If $|\gamma_r|s<1$ for $\alpha_2<s<\alpha_1$ then $A_r/(1-\gamma_r s)$ must be expanded as $A_r(1+\gamma_r s+\gamma_r^2 s^2+\cdots)$. If $|\gamma_r|s>1$ for $\alpha_2<s<\alpha_1$ then $A_r/(1-\gamma_r s)$ must be expanded as

$$-A_r\left(\frac{1}{\gamma_r s}+\frac{1}{\gamma_r^2 s^2}+\cdots\right).$$

We can place γ_1 and γ_2 into the proper categories because $\gamma_1 s=s/\alpha_1<1$ for $\alpha_2<s<\alpha_1$ and $\gamma_2 s=s/\alpha_2>1$ for $\alpha_2<s<\alpha_1$. We can also place the roots γ_r, $r\neq 1, 2$, into the proper categories since from (14) and the fact that $[\beta]_t=\beta_t=\beta_0(1+\phi)^t$ for $t\geqq 0$, there can be only one root γ of $x\theta^p-\theta+1=0$ such that $|\gamma|s<1$ for $\alpha_2<s<\alpha_1$, and that root γ_1 is $1+\phi$.

It should be noticed that the condition $|x|<(p-1)^{p-1}/p^p$, which allows E to be expanded, is also the condition that the series β_t and ϕ should be convergent.

4. **Roots of $x(1+\theta)^{m+n}-\theta^n=0$.** Let $p=(m+n)/n$, where m and n are positive integers, and let ω be an nth primitive root of unity. Writing

$$\nu_r = \phi_p(\omega^r x^{1/n}) = \sum_{j=1}^{\infty} \frac{1}{jp+1}\binom{jp+1}{j}\omega^{rj}x^{j/n}, \tag{15}$$

we have from (5), $\omega^r x^{1/n}(1+\nu_r)^p=\nu_r$ and hence $x(1+\nu_r)^{m+n}=\nu_r^n$. Similarly $x(1+\mu_r)^{m+n}=\mu_r^m$, where, with q denoting $(m+n)/m$,

$$\mu_r = \phi_q(\omega'^r x^{1/m}) = \sum_{j=1}^{\infty} \frac{1}{jq+1}\binom{jq+1}{j}\omega'^{rj}x^{j/m} \tag{16}$$

(ω' being an mth primitive root of unity).

Therefore $x[1+(1/\mu_r)]^{m+n}=(1/\mu_r)^n$. Thus each of the quantities $\nu_0, \nu_1, \cdots, \nu_{n-1}, 1/\mu_0, 1/\mu_1, \cdots, 1/\mu_{m-1}$ satisfies the equation $x(1+\theta)^{m+n}-\theta^n=0$. Since it is clear from (15) and (16) that these quantities are all distinct, it follows that they are the $m+n$ roots of the equation.

As an example, in the case of the quintic $x(1+\theta)^5-\theta=0$, the roots are:

$$v_0 = \phi_5(x) = x + \frac{10}{1.2}x^2 + \frac{15.14}{1.2.3}x^3 + \frac{20.19.18}{1.2.3.4}x^4 + \cdots,$$

$$1/\mu_0 = [\phi_{5/4}(x^{1/4})]^{-1} = x^{-1/4}[1 + \phi_{5/4}(x^{1/4})]^{-5/4},$$

which from (6) is equal to $x^{-1/4}\Phi_{5/4}(x^{1/4})$, where

$$\Phi_{5/4}(x^{1/4}) = 1 - \frac{5}{4}x^{1/4} - \frac{1}{2}\cdot\frac{5}{4}\cdot\frac{1}{4}x^{1/2} - \frac{1}{3}\cdot\frac{5}{4}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}}{1.2}x^{3/4} - \cdots,$$

and

$$1/\mu_1 = [\phi_{5/4}(ix^{1/4})]^{-1} = -ix^{-1/4}\Phi_{5/4}(ix^{1/4}),$$
$$1/\mu_2 = [\phi_{5/4}(-x^{1/4})]^{-1} = -x^{-1/4}\Phi_{5/4}(-x^{1/4}),$$
$$1/\mu_3 = [\phi_{5/4}(-ix^{1/4})]^{-1} = ix^{-1/4}\Phi_{5/4}(-ix^{1/4}).$$

The trinomial equation $z^{\lambda+\mu}+Az^{\mu}+AB=0$ has been solved by Frame [2] in a form which has similarities to our solution of the equation $x(1+\theta)^{m+n}-\theta^n=0$. However, in general our equation is not transformable into Frame's form, although if $n=1$ it becomes (with $1+\theta=z$)

$$z^{m+1} - \frac{1}{x}z + \frac{1}{x} = 0.$$

This is Frame's form with $\lambda=m$, $\mu=1$, $A=-1/x$, $B=-1$.

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A METHOD FOR COMPUTING THE REAL ROOTS OF DETERMINANTAL EQUATIONS

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1. The determinantal equations to be considered are of the form

$$\det (A - \lambda B) = 0, \tag{1}$$

where A, B are real, symmetric, $n \times n$ matrices. These equations arise in the study of undamped small vibrations of a mechanical system about a position of stable equilibrium ([6] Chs. VII, VIII). In this application, A and B are both positive definite. It is shown below that general polynomial equations with real coefficients may be cast in this form, with matrices A and B which are not necessarily definite.

2. Consider first the case in which the matrix B is definite and the roots of (1) are, therefore, necessarily real. For real values of λ the matrix

$$K(\lambda) = A - \lambda B \tag{2}$$

is symmetric. The characteristic value-vector problem for (2) defines characteristic values μ_i and normalized characteristic vectors y_i for which

$$Ky_i = \mu_i y_i \tag{3}$$

and it follows at once that

$$\det (A - \lambda B) = \pi \mu_i, \tag{4}$$

so that λ is a root of (1) when, and only when, one of the functions $\mu_i(\lambda)$ defined by the problem (3) is zero. Moreover, for any real value of λ, the μ_i and the associated vectors y_i may be computed by the Jacobi method of iterated rotations [3][4]. The n roots of (1) are obtained by solving the n equations

$$\mu_i(\lambda) = 0. \tag{5}$$

The Newton-Raphson iteration ([2] Sec. 3.312) may be applied to solve these equations. This application is based upon the formula

$$d\mu_i/d\lambda = -(y_i B y_i) \tag{6}$$

due to Jacobi [4] (see also [1], p. 283) and leads to the iteration

$$\lambda_{k+1} = \phi(\lambda_k), \tag{7}$$

where

$$\phi(\lambda) = (y_i A y_i)/(y_i B y_i). \tag{8}$$

Experiments with a number of cases for which $n \leqq 10$ indicate that, with $\lambda_1 = 0$, convergence of the iteration (7) to ten places is ordinarily obtained in three steps. The amount of computation involved is formidable, however, as the com-

plete solution requires approximately $3n$ applications of the diagonalization procedure.

If $U(\lambda)$ is the orthogonal matrix which diagonalizes $K(\lambda)$, the numerator and denominator of (8) are corresponding diagonal elements of the matrices $U^{-1}AU$ and $U^{-1}BU$ respectively. Thus, during the diagonalization procedure, it is convenient to transform A and B separately and obtain the elements of (8) as by-products of this procedure.

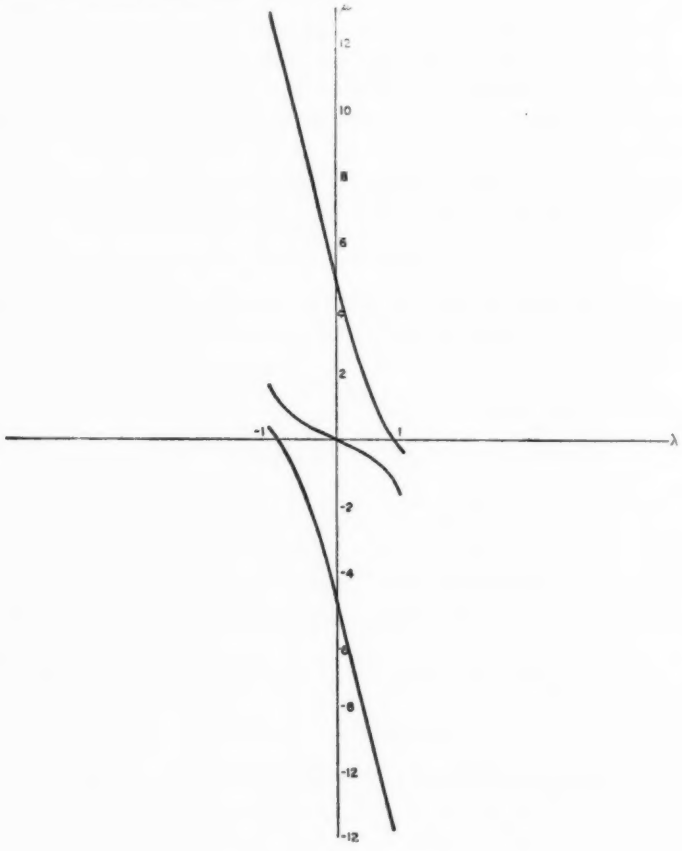

FIG. 1

Formula (6) shows that the functions $\mu_i(\lambda)$ are monotonic, and hence, admit exactly one root apiece. Further information concerning the functions $\mu_i(\lambda)$ may be obtained from the formulae for their second derivatives, which may be written (see [1], *loc. cit.*)

$$\frac{d^2\mu_i}{d\lambda^2} = -2\sum_{k\neq i}\frac{q_{ik}^2}{\mu_i-\mu_k},\quad q_{ik}=\frac{(y_i(BA-AB)y_k)}{\mu_i-\mu_k}.$$

As one might expect, the $\mu_i(\lambda)$ are linear functions of λ, and the y_i are independent of λ, whenever A and B commute.

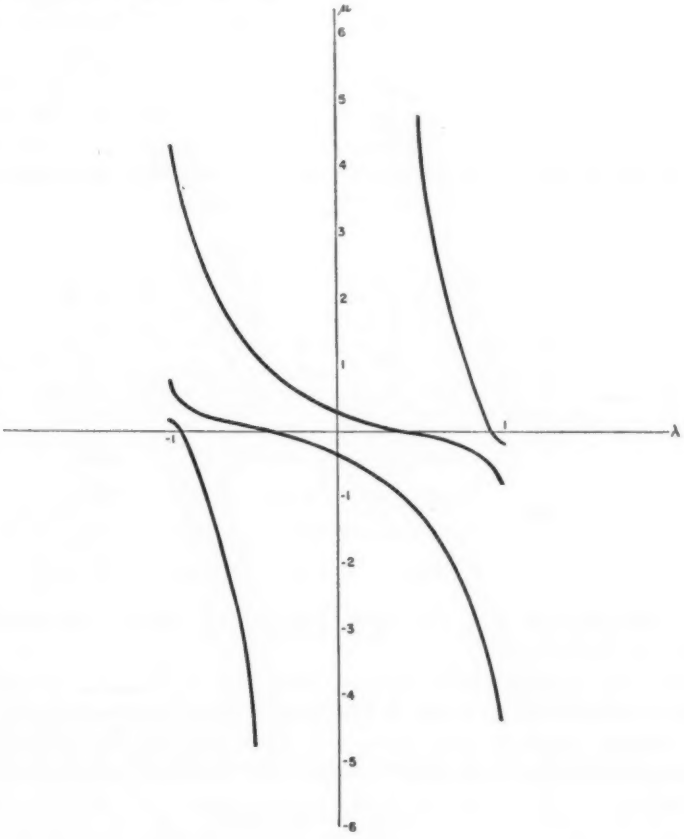

FIG. 2

Figures 1 and 2 are sketches of the functions μ_i obtained by writing the Chebyshev polynomials of the first kind, of degrees three and four, respectively, in the form $T_n(x)\alpha \det(A-xB)$, where B is positive definite (see [5], p. 26). In these cases the elements of $AB-BA$ are somewhat larger than those of either A or B, and the plots indicate the corresponding nonlinear character of these functions.

3. The procedure by which an arbitrary polynomial with real coefficients may be written in the form (1) is illustrated by the case of the polynomial

$$P_4(\lambda) = \lambda^4 - a_1\lambda^3 - a_2\lambda^2 - a_3\lambda - a_4 = 0 \qquad (a_4 \neq 0). \tag{9}$$

The companion matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has the property that $\det(M-\lambda I) \equiv P_4(\lambda)$, may be symmetrized by premultiplication with the nonsingular symmetric matrix

$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & 0 \\ a_3 & a_4 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

For, in this case

$$A = BM = \begin{pmatrix} a_1^2 + a_2 & a_1a_2 + a_3 & a_1a_3 + a_4 & a_1a_4 \\ a_1a_2 + a_3 & a_2^2 + a_4 & a_2a_3 & a_2a_4 \\ a_1a_3 + a_4 & a_2a_3 & a_3^2 & a_3a_4 \\ a_1a_4 & a_2a_4 & a_3a_4 & a_4^2 \end{pmatrix} \tag{11}$$

is clearly symmetric, so that, with these definitions of A and B, the equation (9) is written in the form (1).

4. Formulas (10) and (11) make possible the construction of numerical examples in which neither A nor B is definite. Indeed, any polynomial with a pair of complex conjugate roots must give rise to precisely this situation. The case of the quartic $\lambda^4-1=0$ has been studied both analytically and numerically. In this case

$$AB - BA = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

and $\det(K-\mu I) \equiv (\mu-1+\lambda)(\mu-1-\lambda)(\mu^2-\lambda^2-1)$ so that the loci of the characteristic values μ_i consist in two straight lines $\mu=1-\lambda$, $\mu=1+\lambda$, together with the hyperbola $\mu^2-\lambda^2-1=0$.

Numerical results obtained in this case give values of the μ_i corresponding to the two lines and the two branches of the hyperbola. Convergence to the roots ± 1 was obtained in two iterations, as was expected. In the case of the hyperbola, convergence to the minimum (maximum) point was obtained in three or four iterations, where the denominator of (8) vanished, bringing the iteration to a halt. Thus the procedure would appear to be effective in evaluating the real roots of a polynomial, regardless of the existence of complex roots.

5. More efficient methods than that proposed are available for the solution of determinantal equations in the case in which B, or A, is definite. Thus the proposed method, while believed to be new, is of theoretical interest only, at least in this case. The method may find some practical application, however, as a stable, accurate process for evaluating the real roots of polynomial equations.

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MATHEMATICAL NOTES

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ON THE IDENTITIES OF DIRECT PRODUCTS OF CERTAIN ALGEBRAS

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1. Introduction. In a recent paper [2], the author proved that every strictly functionally complete algebra $\mathfrak{U}$ of more than one element has the property that all the strict identities of $\mathfrak{U}$ are consequences of a certain *finite* subset of these identities. In this present note, we generalize this result to a direct product $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ of the strictly functionally complete algebras $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ where $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ form an independent set of algebras. Indeed, we shall prove the following

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THEOREM. *Let* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be an independent set of strictly functionally complete algebras, and let the order of* $\mathfrak{U}_i = M_i \geqq 2$ $(i = 1, \cdots, n)$. *Let* $\mathfrak{U} = \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *be the direct product of* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$. *Then there exists a finite set* I *of strict identities of* $\mathfrak{U}$ *with the property that every strict identity of* $\mathfrak{U}$ *is a consequence of the identities in* I.

2. Preliminaries. In preparation for the proof of the above theorem, we shall first recall the basic concepts involved (see [1], [2]).

Let $\mathfrak{U} = (U, p, \cdots)$ be a universal algebra with primitive operations, $p, \cdots$. Let $U = \{\cdots, x, \cdots\}$.

A (k-ary) $\mathfrak{U}$-*function* $f(x_1, \cdots, x_k)$ is a composition, via the primitive operations, of indeterminate symbols $x_1, \cdots, x_k$ over the set U together with a (possibly empty) set of constants† $(=\text{fixed} \in U)$.

An $\mathfrak{U}$-function is called a *strict* $\mathfrak{U}$-*function* if it involves no constants. If $\mathfrak{U}$-functions $f(x, \cdots)$ and $g(x, \cdots)$ represent the same mapping we speak of an $\mathfrak{U}$-*identity* $f(x, \cdots) = g(x, \cdots)$. If both f and g are strict $\mathfrak{U}$-functions we speak of a *strict* $\mathfrak{U}$-*identity*. $\mathfrak{U}$ is *finite*, of order n, if U is a class of n elements. $\mathfrak{U}$ is said to be (*functionally*) *complete*—respectively (*functionally*) *strictly complete*—if U is finite and if each mapping of the set $U \times \cdots \times U$ into U may be expressed as some $\mathfrak{U}$-function—respectively as some *strict* $\mathfrak{U}$-*function*.

We now proceed to define the concept of *independence* (see [1]). Let $\text{Sp} = (n_1, n_2, \cdots)$ be a given finitary species, and let $o_1, o_2, \cdots$ denote the primitive operation symbols of this species, where o_i is n_i-ary, $o_i = o_i(x_1, x_2, \cdots, x_{n_i})$. An *expression* $\phi(x, \cdots)$ of species Sp is one or more indeterminate symbols $x, \cdots$ composed via the operation symbols o_i.

In the various universal algebras $\mathfrak{U}_1, \mathfrak{U}_2, \cdots$ of species Sp we shall usually use the same symbols $o_i, \cdots$ to denote the respective primitive operations, *e.g.*, $\mathfrak{U}_1 = (U_1, o_1, o_2, \cdots)$, $\mathfrak{U}_2 = (U_2, o_1, o_2, \cdots)$, etc. Thus, in each $\mathfrak{U} = (U, o_1, o_2, \cdots)$, every expression $\phi(x, \cdots)$ becomes a particular $\mathfrak{U}$-function if we let $x, \cdots$ range over U. We denote this $\mathfrak{U}$-function by $\phi(x, \cdots) (\text{mod } \mathfrak{U})$, or $\phi(x, \cdots)(\mathfrak{U})$. Similarly, $\phi_1 = \phi_2 (\text{mod } \mathfrak{U})$, also written $\phi_1 \equiv \phi_2 (\text{mod } \mathfrak{U})$, denotes that the above is an identity of the algebra $\mathfrak{U}$, when the expressions ϕ_1, ϕ_2 are "interpreted" mod $\mathfrak{U}$.

Let $\{\mathfrak{U}_i\} = \{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ be a finite set of algebras of the same species Sp. We shall say that $\{\mathfrak{U}_i\}$ satisfies the *Chinese residue condition* or $\{\mathfrak{U}_i\}$ is an *independent* set of algebras if, corresponding to each set of expressions $\phi_1, \cdots, \phi_n$, there exists an expression E such that $E = \phi_i (\text{mod } \mathfrak{U}_i)$, for $i = 1, \cdots, n$.

For examples and criteria of independence see [1].

3. The main theorem. For the proof of the main theorem (see introduction), we need the following result which was proved in [1] and which we shall state as a lemma.

LEMMA 1. *Suppose* $\tilde{\mathfrak{U}} = \{\mathfrak{U}_i\} = \{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ *is a set of algebras* $\mathfrak{U}_i$, *each of*

† A constant may also be defined as a "0-ary" function.

which is strictly functionally complete and of order >1, *and where the algebras* $\mathfrak{U}_i$ *are of the same species* Sp. *Suppose that* $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ *form an independent set of algebras. Suppose that* $\overline{\mathfrak{U}}$ *is an algebra of species* Sp. *A sufficient (and necessary) condition for* $\overline{\mathfrak{U}}$ *to be subdirectly representable in* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *is that* $\overline{\mathfrak{U}}$ *satisfies all strict identities which are common to the algebras* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *or, what is the same, that* $\overline{\mathfrak{U}}$ *satisfies all strict identities of the algebra* $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *(direct product).*

This is essentially a reformulation of Theorem 6.1 (A) of [1].

Although in the above lemma we assumed that *each* strict identity of $\mathfrak{U}_1 \times \mathfrak{U}_2 \times \cdots \times \mathfrak{U}_n$ is also an identity of $\overline{\mathfrak{U}}$, in the proof of this lemma only a *finite* number of strict $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$-identities were assumed to be $\overline{\mathfrak{U}}$-identities (see [1]). Let this *finite* set of strict $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$-identities be denoted by I. It is now easily seen that the above lemma can be strengthened as follows:

LEMMA 1′. *Suppose* $\widetilde{\mathfrak{U}}$; $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$; $\overline{\mathfrak{U}}$, *are as in Lemma* 1. *A sufficient (and necessary) condition for* $\overline{\mathfrak{U}}$ *to be subdirectly representable in* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *is that* $\overline{\mathfrak{U}}$ *satisfies all strict identities of* $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *in the finite set* I.

We are now in a position to prove the following

THEOREM 1. *Let* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be a finite independent set of algebras, each of which is strictly functionally complete and of order* >1 *and let all the algebras* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be of the same species* Sp. *Let* $\mathfrak{U} = \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *be the direct product of* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$. *Then the strict identities of* $\mathfrak{U}$ *have the above finite set* I *of strict* $\mathfrak{U}$*-identities as a finite basis, i.e., every strict identity of* $\mathfrak{U}$ *is a consequence of the identities in* I.

Proof. Let B be any algebra of species Sp which satisfies set I, and let L be any strict identity of $\mathfrak{U}(=\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n)$. Theorem 1 will be proved by showing that L is also a strict identity of B. Now, by Lemma 1′, B is subdirectly representable in $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *i.e.*,

(1) $B \cong$ subdirect product of subdirect powers $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$ of $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$.

But, since L is an identity of $\mathfrak{U}_i$ $(i=1, \cdots, n)$, therefore, L is also an identity of the subdirect power $\mathfrak{U}_i^{(N_i)}$ of $\mathfrak{U}_i$ (see [1], Th. 1.1), $(i=1, \cdots, n)$. Hence L is an identity of the direct product of $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$. Therefore, *a fortiori* L is an identity of the subdirect product of $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$. Hence, by (1) above, L is an identity of B. This proves the theorem.

In conclusion, I wish to express my indebtedness and gratitude to Professor A. L. Foster for his generous counsel.

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AN APPLICATION OF THE GENERATING FUNCTION TO DIFFERENTIAL EQUATIONS

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The calculus discussed here is based upon a power series transform for sequences which was introduced by Laplace in 1812 [4], a year before he introduced his now well-known integral transform. This series transform has been extensively studied as a *generating function* of a given sequence (see Erdélyi *et al.* [1], pp. 228–282). A discussion of its applications in the theory of finite differences is found in Jordan ([3], pp. 20–44). Goldberg ([2], pp. 189–207) discusses the application of this transform to the solution of linear difference equations with constant coefficients. First, we will briefly outline some of the basic theory of this transform and its inverse. We will then show how the inverse transform can be applied to formalize the finding of the power series expansion of the solution of a linear differential equation in a manner that seems to be new.

Let $\mathfrak{F}$ denote the set of complex valued functions, defined and holomorphic on some open connected subset (depending upon the individual function) of the complex plane containing the origin. Let $\mathfrak{R}$ denote the set of equivalence classes of $\mathfrak{F}$ so defined that two elements of $\mathfrak{F}$ are equivalent if their restrictions to some neighborhood of zero are identical. For convenience sake, we will not distinguish between an element of $\mathfrak{F}$ and the equivalence class to which it belongs. Now let $\mathfrak{S}$ denote the set of all sequences $a=\{a_n\}$ such that $\overline{\lim}\sqrt[n]{|a_n|}$ is finite. We then consider the mapping $G: \mathfrak{S}\to\mathfrak{R}$ for which $u=G(a)$ if

$$u(t) = \sum_{n=0}^{\infty} a_n t^n$$

for $|t| < (\lim \sqrt[n]{|a_n|})^{-1}$. The function $u=G(a)$ is called the *generating function* of the sequence a. It follows from the theory of the Taylor series that this mapping $G: \mathfrak{S}\to\mathfrak{R}$ is one-to-one; and for the inverse mapping, G^{-1}, $a=G^{-1}(u)$ if and only if

$$a_n = u^{(n)}(0)/n! \tag{1}$$

for $n=0, 1, \cdots$. It is immediate that for $a\in\mathfrak{S}$, $a=G^{-1}(G(a))$; while for $u\in\mathfrak{R}$, $u=G(G^{-1}(u))$.

It follows immediately from their definition that the mappings G and G^{-1} are linear. That is, for α and β constants: if $a\in\mathfrak{S}$, $b\in\mathfrak{S}$, then $(\alpha a+\beta b)\in\mathfrak{S}$ and

$$G(\alpha a + \beta b) = \alpha G(a) + \beta G(b); \tag{2}$$

if $u\in\mathfrak{R}$, $v\in\mathfrak{R}$, then $(\alpha u+\beta v)\in\mathfrak{R}$ and

$$G^{-1}(\alpha u + \beta v) = \alpha G^{-1}(u) + \beta G^{-1}(v). \tag{3}$$

Define the mapping $E^m: \mathfrak{S}\to\mathfrak{S}$, $m=0, 1, \cdots$, by $(E^m a)_n = a_{n+m}$ for $n=0, 1, \cdots$, where $E^1=E$. It can be established by induction that if $U_m=G(E^m a)$, then

$$U_m(t) = t^{-m}[u(t) - a_0 - ta_1 - \cdots - t^{m-1}a_{m-1}] \tag{4}$$

for $t \neq 0$, and $m = 1, 2, \cdots$.

Upon checking the above reference to Goldberg ([2], pp. 189–207), one finds that (2), (3), and (4) are the basis for the operational calculus which makes the generating function applicable for the solution of difference equations. As with the Laplace transform, a table of generating functions is a valuable tool for such applications. Any table of power series expansions can be used as a starting point for such a table, and further entries with their derivations can be found in the references cited above.

We shall now show how the inverse transform, $G^{-1}: \mathfrak{R} \to \mathfrak{S}$, is applicable to the solution of differential equations.

It follows from the Cauchy product theorem for power series that if $u \in \mathfrak{R}$, $v \in \mathfrak{R}$, then $uv \in \mathfrak{R}$ and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv))_n = \sum_{j=0}^{n} a_j b_{n-j} \tag{5}$$

for $n = 0, 1, \cdots$. We might well call the sequence defined by (5) the *convolution* of the sequences a and b, and if we were to denote it by $a*b$, we would then have $G(a)G(b) = G(a*b)$.

It follows from (1) that if $u \in \mathfrak{R}$, then $u^{(1)} \in \mathfrak{R}$, and

$$(G^{-1}(u^{(1)}))_n = \frac{u^{(n+1)}(0)}{n!} = (n+1)\frac{u^{(n+1)}(0)}{(n+1)!} = (n+1)(EG^{-1}(u))_n$$

for $n = 0, 1, \cdots$. We can now establish by induction that $u^{(m)} \in \mathfrak{R}$ and

$$(G^{-1}(u^{(m)}))_n = (n+1)(n+2)\cdots(n+m)(E^m G^{-1}(u))_n \tag{6}$$

for $m = 1, 2, \cdots$, $n = 0, 1, \cdots$. Finally, we combine (5) and (6) to conclude that if $u \in \mathfrak{R}$, $v \in \mathfrak{R}$, then $uv^{(m)} \in \mathfrak{R}$, and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv^{(m)}))_n = \sum_{j=0}^{n} a_j(n-j+1)(n-j+2)\cdots(n-j+m)b_{n-j+m} \tag{7}$$

for $n = 0, 1, \cdots$ and $m = 1, 2, \cdots$.

Consider now the differential equation $(1+t^2)v'' + 2tv' - 2v = 0$. If we define $u_0(t) = (1+t^2)$, $u_1(t) = 2t$, $u_2(t) = -2$, and apply (3), we get

$$G^{-1}(u_0v'') + G^{-1}(u_1v') + G^{-1}(u_2v) = 0. \tag{8}$$

Substituting into (7) with $m = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$, and $a_j = 0$ for $j > 2$, it follows that if $G^{-1}(v) = b$, then

$$(G^{-1}(u_0v''))_0 = 2b_2, \quad (G^{-1}(u_0v''))_1 = 2 \cdot 3b_3,$$

$$(G^{-1}(u_0v''))_n = (n+1)(n+2)b_{n+2} + (n-1)nb_n \qquad \text{for } n \geqq 2.$$

Likewise, we find

$$(G^{-1}(u_1v'))_0 = 0,$$
$$(G^{-1}(u_1v'))_n = 2nb_n \qquad \text{for } n \geqq 1,$$
$$(G^{-1}(u_2v))_n = -2b_n \qquad \text{for } n \geqq 0.$$

Substituting into (8) and simplifying we get the difference system

$$b_2 = b_0,\ b_3 = 0, \qquad (n+1)b_{n+2} + (n-1)b_n = 0, \qquad \text{for } n \geqq 2.$$

It follows that b_0 and b_1 are arbitrary, $b_{2k+1} = 0$, and $b_{2k} = [(-1)^{k+1}/(2k-1)]b_0$ for $k = 1, 2, \cdots$. That is,

$$v(t) = b_0 + b_1t + b_0\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} t^{2k}\right] = b_1t + b_0(1 + t\cdot\tan^{-1} t).$$

The above steps are precisely those met in the usual text-book solution of the differential equation (8) when one assumes a solution of the form

$$v(t) = \sum_{n=0}^{\infty} b_n t^n.$$

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AN ALGEBRAIC PROOF OF KIRCHHOFF'S NETWORK THEOREM*

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1. Introduction. Weyl [10] gave probably the first complete proof of the existence and uniqueness of currents in a resistive network subject to Ohm's law and Kirchhoff's voltage and current laws (Eckmann [2, 3], Roth [7, 8]). Much earlier Kirchhoff [5] gave an explicit expression for these currents in terms of maximal trees of the correlated graph. There are a number of papers on this theorem scattered through the literature, notably those of Ahrens [1] and Franklin [4]; and there has been considerable interest recently in its possible use in connection with computation of solutions to network problems (MacWilliams [6]). We give a simple linear algebraic proof of this ancient theorem, and incidentally obtain for graphs an expression for the projection of a real 1-chain on the space of 1-cycles in terms of maximal trees.

2. Preliminaries. A (finite oriented) graph G consists of a finite set V (of *vertices*) and a subset G of $V \times V$ (of *branches*) such that if $(v_1, v_2) \in G$ then

* This research was supported in part by the U. S. Air Force under Contract No. AF33(616)-2797, monitored by the Aeronautical Research Laboratory, Wright Air Development Center.

$(v_2, v_1) \notin G$. A *subgraph* G_1 of G consists of a subset G_1 of G together with the set V_1 of all vertices occurring on branches of G_1. The space C_1 is the real inner product space having orthonormal basis G; the space C_0 is the real inner product space having orthonormal basis V. We denote inner product by $(\ ,\ )$. The space Z_1 of 1-*cycles* is the null space of the linear transformation $\partial\colon C_1 \to C_0$ having $\partial(v_i, v_j) = v_j - v_i$. Vertices v_i and v_j are *connected* if there is a $c \in C_1$ with $\partial c = v_j - v_i$. G is *connected* if every pair of its vertices is connected. (Note that if $\partial c = v_j - v_i$ there is a c_1 with $\partial c_1 = v_j - v_i$ and $(c_1, b) = 0$ or $(c_1, b) = \pm 1$ for each branch b.) B_1, the space of 1-*coboundaries*, is the orthogonal complement of Z_1. A *forest* is a graph whose space of 1-cycles is 0-dimensional; a *tree* is a connected forest. (Rephrasing the last parenthetical observation, in a tree there is a unique c with $\partial c = v_j - v_i$, and its nonzero coefficients are all ± 1.) Throughout this note $R\colon C_1 \to C_1$ is a linear transformation such that for each branch b, $Rb = r_b b$ for some positive real r_b. If F is a maximal subforest of G, set $W_F = 1$ if $F = G$, $W_F = \prod_{b \in G - F} r_b$ otherwise. If $b \notin F$, $F \cup \{b\}$ contains a unique 1-cycle in which b occurs with coefficient $+1$; let $\overline{F}\colon C_1 \to C_1$ be the linear transformation such that $\overline{F}b$ is this unique 1-cycle if $b \notin F$, $\overline{F}b = 0$ otherwise.

3. Results

THEOREM. *Let G be a connected graph. Define a linear transformation $S\colon C_1 \to C_1$ by $S = \sum_T W_T \overline{T}$, where the summation is over all maximal trees of G. Then RS is self-adjoint.*

LEMMA 1. *Suppose T is a maximal tree. Then if $b_1, \cdots, b_k$ constitute all complementary branches, $\{\overline{T}b_1, \cdots, \overline{T}b_k\}$ is a basis for Z_1.*

Proof. It is clear that the $\overline{T}b_i$ are linearly independent. If z is any 1-cycle, we show that $z = \sum_i (z, b_i)\overline{T}b_i$. For $b_j \notin T$, $(z - \sum_i (z, b_i)\overline{T}b_i, b_j) = (z, b_j) - \sum_i (z, b_i)(\overline{T}b_i, b_j) = (z, b_j) - \sum_i (z, b_i)\delta_{ij} = (z, b_j) - (z, b_j) = 0$. Hence $z - \sum_i (z, b_i)\overline{T}b_i$ is a 1-cycle of T, therefore 0.

COROLLARY. *For any 1-cycle z, $\overline{T}z = z$; for any coboundary b, $\overline{T}^* b = 0$ (where $\overline{T}^*$ denotes the adjoint of $\overline{T}$).*

LEMMA 2. *All nonzero coefficients of $\overline{T}b$ are ± 1.*

Proof. If $b = (v_1, v_2)$, $\partial(\overline{T}b - b) = \partial\overline{T}b - \partial b = -\partial b = v_2 - v_1$.

LEMMA 3. *Suppose T is a maximal tree, that b_i, b_j are branches with $(\overline{T}b_i, b_j) \neq 0$. Then $U = (T - \{b_j\}) \cup \{b_i\}$ is a maximal tree.*

Proof. The only nonzero 1-cycle of $U \cup \{b_j\}$ contains both b_i and b_j. Since U does not contain b_j, it does not contain any nonzero 1-cycle. Maximality follows from Lemma 1.

LEMMA 4. *Suppose T, U, b_i, b_j are as in Lemma 3. Then*

$$(\overline{T}b_i, b_j) = (\overline{U}b_j, b_i), \qquad r_j W_T = r_i W_U.$$

LEMMA 5. *Let X_{ij} be the set of all maximal trees T containing b_j such that $(\overline{T}b_i, b_j) \neq 0$. Then for $i \neq j$,*

$$\sum_{T \in X_{ij}} W_T r_j(\overline{T}b_i, b_j) = \sum_{U \in X_{ji}} W_U r_i(\overline{U}b_j, b_i).$$

Proof of theorem. For $i \neq j$,

$$(RSb_i, b_j) = \sum_T W_T r_j(\overline{T}b_i, b_j) = \sum_{T \in X_{ij}} W_T r_j(\overline{T}b_i, b_j) = \sum_{U \in X_{ji}} W_U r_i(\overline{U}b_j, b_i)$$
$$= \sum_U W_U r_i(\overline{U}b_j, b_i) = (Sb_j, Rb_i) = (S^*Rb_i, b_j).$$

Thus $RS = S^*R = (RS)^*$ and RS is self-adjoint.

COROLLARY. *Let N be the number of maximal trees of G. Then $(1/N)\sum_T \overline{T}$ is the orthogonal projection of C_1 on Z_1.*

Proof. By the corollary to Lemma 1, $(1/N)\sum_T \overline{T}$ is the identity on Z_1; by the theorem, it is self-adjoint. Thus for a 1-coboundary b,

$$\left(\frac{1}{N}\sum_T \overline{T}b, b_i\right) = \left(b, \frac{1}{N}\sum_T \overline{T}b_i\right) = 0.$$

COROLLARY. *For $c \in C_1$, there is exactly one $z \in Z_1$ with $Rz - c \in B_1$. Moreover, $z = (1/\Delta)R^{-1}S^*c$, where $\Delta = \sum_T W_T$.*

Proof. For any such z, $S^*(Rz - c)$ is in B_1 and hence in the null space of each term of $S^* = \sum_T W_T \overline{T}^*$. Since each $\overline{T}$ is the identity on Z_1 we have

$$S^*c = S^*Rz = RSz = R\sum_T W_T \overline{T}z = R\sum_T W_T z = R\Delta z = \Delta Rz,$$

from which existence, uniqueness, and the asserted formula follow. (For a discussion of N and Δ see Trent [9].)

The first and second sentences of this corollary are respectively the previously mentioned theorems of Weyl and Kirchhoff. In network terminology, G is a network with resistance r_b on branch b and voltage source (c, b) on branch b; (z, b) is the current resulting in branch b (see also Eckmann [2] or Roth [7]). By taking inner products with a branch b, one obtains

$$(z, b) = \frac{1}{\Delta}\sum \frac{W_T}{r_b}(c, \overline{T}b),$$

where the summation extends over all maximal trees T such that $b \in G - T$. This is the original form of Kirchhoff's result. The conditions $z \in Z_1$ and $Rz - c \in B_1$ express Ohm's law and Kirchhoff's voltage and current laws.

We omit consideration of other coefficient fields; for example, complex numbers in the case of lumped parameter networks ([2], [7]).

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THE RECURRENCE THEOREM

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The purpose of this note is to give a completely self-contained account of the recurrence theorem of ergodic theory. This theorem was first proved by Poincaré, and, in modern form, asserts that if T is a measure-preserving transformation on a finite measure space and if E is any measurable set, then almost every point of E returns to E infinitely often under application of T. More generally, if all of the transformations T, T^2, T^3, $\cdots$, are incompressible, the same conclusion holds for any measure space. In 1947, Halmos [1] showed that if T is incompressible, one-one, and if T^{-1} is measurable, then all powers of T are incompressible. In 1959, Taam [2] succeeded in removing the restrictions on T, by carefully analyzing the already quite involved combinatorial proof of Halmos. Independently, in 1959 the author of this note, in the course of an investigation of the properties of endomorphisms of Boolean algebras, discovered a very simple proof of the recurrence theorem which circumvents the necessity of proving that powers are incompressible. This proof appears midway in a paper [3] which introduces considerable machinery whose function is more general than a proof of the recurrence theorem, and consequently it appears more difficult than it is, in fact. We show here that this theorem is completely on the surface.

Throughout, let X be a set, let $\mathcal{S}$ be a σ-algebra of subsets of X, and let $\mathcal{I}$ be a σ-ideal of $\mathcal{S}$. The triple $(X, \mathcal{S}, \mathcal{I})$ will be called a *measurability space.* A

* Research fellow of the Alfred P. Sloan Foundation.

mapping T of X into itself will be called a *measurable* transformation if $E\in\mathcal{S}$ implies $T^{-1}E\in\mathcal{S}$. If T is a measurable transformation and if $E\in\mathcal{S}$, the set $E-\bigcup_{j=1}^{\infty} T^{-j}E$ is the set of all those points in E which never return to E, and the set $E-\bigcap_{m=0}^{\infty}\bigcup_{j=m+1}^{\infty} T^{-j}E$ is the set of those points of E which do not return infinitely often to E. The measurable transformation T is called *recurrent* if $E-\bigcup_{j=1}^{\infty} T^{-j}E\in\mathcal{I}$ for all $E\in\mathcal{S}$, and is called *infinitely recurrent* if $E-\bigcap_{m=0}^{\infty}\bigcup_{j=m+1}^{\infty} T^{-j}E\in\mathcal{I}$ for all $E\in\mathcal{S}$.

A measurable transformation T is called *incompressible* if $E\subset T^{-1}E$ implies $T^{-1}E-E\in\mathcal{I}$. We remark that T is incompressible if and only if $T^{-1}E\subset E$ implies $E-T^{-1}E\in\mathcal{I}$. For, let $E\in\mathcal{S}$ and let $F=X-E$. Then $E\subset T^{-1}E$ if and only if $T^{-1}F\subset F$, and $E-T^{-1}E=T^{-1}F-F$, $T^{-1}E-E=F-T^{-1}F$.

A set $E\in\mathcal{S}$ is called a *wandering set* if the sequence $E, T^{-1}E, T^{-2}E, \cdots$, is disjoint. A measurable transformation is called *conservative* if every wandering set is in $\mathcal{I}$.

THE RECURRENCE THEOREM. *Let T be a measurable transformation of a measurability space $(X, \mathcal{S}, \mathcal{I})$. Then the following are equivalent*: (1) *T is incompressible*; (2) *T is conservative*; (3) *T is recurrent*; (4) *T is infinitely recurrent.*

Proof. For any $E\in\mathcal{S}$, let $E^*=\bigcup_{j=1}^{\infty} T^{-j}E$. We first prove that (1) implies (3). Let $E\in\mathcal{S}$, and let $A=E\cup E^*$. Then $T^{-1}A=E^*\subset A$, and since T is incompressible, $A-T^{-1}A\in\mathcal{I}$. But $A-T^{-1}A=E-E^*$.

Next, (3) implies (2). Suppose E is a wandering set; then E and E^* are disjoint, so that $E-E^*=E$. If T is recurrent, then $E=E-E^*\in\mathcal{I}$, so that T is conservative.

Now we show that (2) implies (1). If E is any set in $\mathcal{S}$, then $E-E^*$ is a wandering set. If T is conservative, $E-E^*\in\mathcal{I}$. But then if $T^{-1}E\subset E$, we have $E^*=T^{-1}E$, and hence $E-T^{-1}E\in\mathcal{I}$. Thus T is incompressible.

Finally, we prove that (4) and (3) are equivalent. First observe that if $B=\bigcap_{m=0}^{\infty} T^{-m}E^*$, then $B\subset E^*$ and $E-E^*\subset E-B$. Thus if T is infinitely recurrent, it is recurrent. Conversely, assume (3). By what has been shown, we know that (1) holds; we use both (1) and (3). It is clear that $E-B=(E-E^*)\cup[E\cap\bigcup_{m=1}^{\infty}(T^{-m}E^*-T^{-(m+1)}E^*)]$. Since T is recurrent, $E-E^*\in\mathcal{I}$. Since $T^{-1}(T^{-m}E^*)=T^{-(m+1)}E^*\subset T^{-m}E^*$ for each $m\geq 0$, and since T is incompressible, $T^{-m}E^*-T^{-(m+1)}E^*\in\mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal, then $E-B\in\mathcal{I}$. This completes the proof.

We must remark that in the proof of the equivalence of (1), (2), and (3), *no use was made of any property whatsoever of the family $\mathcal{I}$.*

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WEDDERBURN'S THEOREM AND A THEOREM OF JACOBSON

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In teaching an undergraduate class in modern algebra (whose students, although very bright, were mostly sophomores and so had little algebraic knowledge or technique) the author was faced with the problem of presenting to the class two theorems which long had been among his favorites, namely, Wedderburn's theorem on finite division rings and (in the division ring case) Jacobson's theorem that a ring in which $x^{m(x)}=x$ for all x is commutative. In order to do so he devised the proofs presented here; these proofs may be of interest to others confronted with similar problems.

The only facts needed to follow the proofs and which are not absolutely trivial are:

1. The multiplicative group of a finite field is cyclic.

2. If F is a finite field, $\alpha\neq 0\in F$ then there exist λ, $\mu\in F$ so that $1+\lambda^2-\alpha\mu^2=0$.

Of course, many proofs of Wedderburn's theorem exist. In fact, we presented Wedderburn's original proof [2] and the slight twist, in its "punch-line," introduced by Witt [3]. Other proofs, to name but a few, are those of Artin [1] and Zassenhaus [4]. The proof here is closest in spirit to that of Artin, but seems to be both shorter and more elementary. Moreover, no use is made either of counting or of nontrivial number theory. It is of interest that the proof finally hinges on the fact that the quaternions over a finite field do not form a division ring. This is equivalent to making use of fact (2) in the introduction.

We begin with

LEMMA 1. *Let D be a division ring of characteristic $p>0$ with center Z, and P the prime field with p elements contained in Z. Suppose $a\in D$, $a\notin Z$ is such that $a^{p^n}=a$ for some $n>0$. Then there exists an $x\in D$ such that*

(1) $xax^{-1}\neq a$,

(2) $xax^{-1}\in P(a)$, *the field obtained by adjoining a to P.*

Proof. Define the mapping $\delta: D\to D$ by $\delta(x)=xa-ax$ for all $x\in D$. $P(a)$ is a finite field, and has, say, p^m elements. These all satisfy the equation $u^{p^m}=u$. By a trivial verification we immediately have that $\delta^p(x)=xa^p-a^px$, $\delta^{p^m}(x)=xa^{p^m}-a^{p^m}x=xa-ax=\delta(x)$ for all $x\in D$. Thus $\delta^{p^m}=\delta$.

Now if $\lambda\in P(a)$, $\delta(\lambda x)=(\lambda x)a-a(\lambda x)=\lambda(xa-ax)=\lambda\delta(x)$ since λ commutes with a. Thus if I denotes the identity map on D, $\delta\lambda I=(\lambda I)\delta$ for all $\lambda\in P(a)$. Now the polynomial $u^{p^m}-u$ considered over $P(a)$ has all its p^m roots as the elements of $P(a)$. Thus $u^{p^m}-u=\prod_{\lambda\in P(a)}(u-\lambda)$. Thus since δ commutes with all λI, $\lambda\in P(a)$,

$$0=\delta^{p^m}-\delta=\prod_{\lambda\in P(a)}(\delta-\lambda I).$$

If for every $\lambda\neq 0\in P(a)$, $\delta-\lambda I$ annihilates no nonzero element in D, then since

$\delta(\delta-\lambda_1 I)\cdots(\delta-\lambda_k I)=0$ we would get that $\delta=0$, that is that $xa-ax=0$ for all $x\in D$, forcing $a\in Z$ contrary to hypothesis. Thus there is a $\lambda\neq 0\in P(a)$ and an $x\neq 0\in D$ so that $(\delta-\lambda I)x=0$; that is $xa-ax=\lambda x$ and so $xax^{-1}=a+\lambda\neq a\in P(a)$, proving the lemma.

COROLLARY. *In Lemma* 1 $xax^{-1}=a^i\neq a$ *for some integer* i.

Proof. Let a be of order s, then in the field $P(a)$ all the roots of the polynomial u^s-1 are $1, a, a^2, \cdots, a^{s-1}$. Since xax^{-1} is in $P(a)$ and is a root of this polynomial, $xax^{-1}=a^i$ follows.

We first prove the

THEOREM (Wedderburn). *A finite division ring is a field.*

Proof. Let D be a finite division ring. We assume the theorem to be true for division rings with fewer elements than D.

We first remark that if $a, b\in D$ are such that $b^ta=ab^t$ but $ab\neq ba$, then $b^t\in Z$. For consider $N(b^t)=\{x\in D \mid xb^t=b^tx\}$. $N(b^t)$ is a subdivision ring of D, so if it were not D it would be commutative. Since $a, b\in N(b^t)$ and these do not commute, it must then be that $N(b^t)=D$.

Pick $a\in D$, $a\notin Z$ such that a minimal positive power of a falls in Z. Clearly this minimal power is a prime, r. By the corollary to Lemma 1, there is an $x\in D$ such that $xax^{-1}=a^i\neq a$. Since r is a prime, using the little Fermat theorem, $x^{r-1}ax^{-(r-1)}=a^{i^{r-1}}=a^{1+ru}=aa^{ru}=\lambda a$ where $a^{ru}=\lambda\in Z$. Since $x\notin Z$, $x^{r-1}\notin Z$ by the minimal nature of r; thus by the remark in the paragraph above, $x^{r-1}a\neq ax^{r-1}$, so that $\lambda\neq 1$. Let $b=x^{r-1}$. Thus $bab^{-1}=\lambda a$; consequently $a^r=ba^rb^{-1}=(bab^{-1})^r=\lambda^r a^r$, forcing $\lambda^r=1$. We claim that if $y\in D$ is such that $y^r=1$ then $y=\lambda^i$, for in the field $Z(y)$ there are at most r roots of the polynomial t^r-1 and these are already given by the powers of λ. Now $b^r=\lambda^r b^r=(a^{-1}ba)^r=a^{-1}b^ra$; thus $b^ra=ab^r$, $ba\neq ab$ which implies that $b^r\in Z$. The multiplicative group of Z is cyclic and is generated by an element γ. Thus $a^r=\gamma^n$, $b^r=\gamma^m$. If $n=kr$ then $(a/\gamma k)^r=1$, which would make $a/\gamma k=\lambda^i$ and would lead to $a\in Z$. Thus $r\nmid n$; similarly $r\nmid m$. Let $a_1=a^m$, $b_1=b^n$. Thus $a_1^r=a^{mr}=\gamma^{mn}=b^{nr}=b^r$ and

$$a_1b_1=\mu b_1a_1(\mu\neq 1\in Z) \tag{1}$$

($\mu\neq 1$ since $r\nmid n, m$ so $a_1, b_1\notin Z$) and $\lambda^r=1$. Computing $(b_1^{-1}a_1)^r$ using (1) we arrive at

$$(b_1^{-1}a_1)^r=\mu^{-(1+2+\cdots+(r-1))}b_1^{-r}a_1^r=\mu^{-r(r-1)/2}.$$

If r is odd then since $\mu^r=1$, we have that $(b_1^{-1}a_1)^r=1$. But then $b_1^{-1}a_1=\lambda^i$; and this implies that $a_1b_1=b_1a_1$, a contradiction. Hence if r is odd the theorem is proved.

If $r=2$, then since $\mu^2=1$, $\mu\neq 1$ we have $\mu=-1$. The characteristic must also then be different from 2. Also $\alpha=a_1^2=b_1^2\in Z$, $a_1b_1=-b_1a_1\neq b_1a_1$. In Z there are elements ξ, η so that $1+\xi^2-\alpha\eta^2=0$. But then $(a_1+\xi b_1+\eta a_1b_1)^2=\alpha(1+\xi^2-\alpha\eta^2)$

$=0$. Being in a division ring this yields $a_1+\xi b_1+\eta a_1 b_1=0$. Thus

$$0 \neq 2a_1^2 = a_1(a_1 + \xi b_1 + \eta a_1 b_1) + (a_1 + \xi b_1 + \eta a_1 b_1)a_1 = 0.$$

This contradiction finishes the proof.

We now proceed to prove the

THEOREM (Jacobson). *Let D be a division ring such that for every $x\in D$ there exists an integer $n(x)>1$ so that $x^{n(x)}=x$. Then D is commutative.*

Proof. For $a\neq 0\in D$, $a^n=a$, $(2a)^m=2a$. Putting $q=(n-1)(m-1)+1$ we have $q>1$, $a^q=a$, $(2a)^q=2a$, so that $(2^q-2)a=0$. Thus D has characteristic $p>0$. If P is the prime field with p elements contained in Z, then $P(a)$ has p^h elements, so that $a^{p^h}=a$. So if $a\notin Z$ the conditions of Lemma 1 are satisfied and so there exists a $b\in D$ such that (1) $bab^{-1}=a^u\neq a$. Suppose $b^{p^k}=b$ and consider

$$W = \left\{x \in D \,\middle|\, x = \sum_{i,j}^{p^h,p^k} p_{ij}a^i b^j,\ p_{ij} \in P\right\}.$$

W is finite, is closed under addition, and by virtue of (1) is closed under multiplication. Thus W is a finite ring and being a subring of the division ring D the two cancellation laws hold so it is a finite division ring. But then it is commutative. Since a, $b\in W$ this forces $ab=ba$, contrary to $a^ub=ba$. This proves the theorem.

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A NOTE ON THE GENERALIZED WILSON'S THEOREM

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It is familiar that if $m\geqq 1$ and W_m denotes the product of the integers $\leqq m$ and prime to m then ([2], Ths. 47, 59)

$$W_m \equiv \pm 1 \pmod{m}, \tag{1}$$

where the lower sign occurs provided $m=1, 2, 4, p^r, 2p^r$ ($r\geqq 1$) and p is a prime >2. (For other references see [1], Ch. 3.)

In this note we prove the following related result.

Let p be a fixed prime >3 and let P_m denote the product of the integers $\leqq m$ and prime to p. Then if $p^r \mid m$, $r\geqq 1$, we have

$$P_m \equiv ((p-1)!)^{m/p} \pmod{p^{r+2}} \qquad (p>3). \tag{2}$$

Proof. Put $m=np$. Then

$$P_{np} = \prod_{h=1}^{n} \prod_{k=1}^{p-1} (hp - k). \tag{3}$$

If we put

$$(x-1)(x-2)\cdots(x-p+1) = x^{p-1} - A_1 x^{p-2} + \cdots + A_{p-1},$$

then it is known that ([3] p. 326)

$$A_{2r} \equiv 0 \pmod{p}, \qquad A_{2r+1} \equiv 0 \pmod{p^2} \qquad (1 \leqq r < \tfrac{1}{2}(p-1)).$$

It follows that

$$\binom{hp-1}{p-1} = 1 + c_1 p^3 h + c_2 p^3 h^2 + c_3 p^5 h^3 + c_4 p^5 h^4 + \cdots + c_{p-1} p^{p-1} h^{p-1}, \tag{4}$$

where the c_j are rational numbers that are integral (mod p).

In particular (4) implies

$$\binom{hp-1}{p-1} \equiv 1 \pmod{p^3},$$

so that $P_{np} \equiv ((p-1)!)^n \pmod{p^3}$. This is the case $r=1$ of (2).

For the general case we must show that

$$\prod_{h=1}^{p^r} f(h) \equiv 1 \pmod{p^{r+2}}, \tag{5}$$

where $f(h)$ denotes the right member of (4).

We prove (5) by induction on r. Clearly

$$\prod_{h=1}^{p^r} f(h) = \prod_{k=0}^{p-1} \prod_{h=1}^{p^{r-1}} f(kp^{r-1} + h). \tag{6}$$

Since

$$f(kp^{r-1} + h) \equiv f(h) + (c_1 + 2c_2 h) k p^{r+2} \pmod{p^{r+3}},$$

it follows that

$$\prod_{k=0}^{p-1} f(kp^{r-1} + h) \equiv (f(h))^p \pmod{p^{r+3}}.$$

By the inductive hypothesis

$$\prod_{h=1}^{p^{r-1}} (f(h))^p = (1 + p^{r+2} M)^p \equiv 1 \pmod{p^{r+3}}, \tag{7}$$

where M is integral (mod p). Hence by (6) and (7)

$$\prod_{h=1}^{p^r} f(h) \equiv 1 \pmod{p^{r+3}}.$$

This completes the proof of (2).

As a special case of (2) we note that

$$P_{p^r} \equiv ((p-1)!)^{p^{r-1}} \pmod{p^{r+2}} \qquad (p > 3, r \geqq 1). \tag{8}$$

Since

$$Q_r = \frac{p^r!}{p^{r-1}!(p!)^{p^{r-1}}} = \frac{P_r}{((p-1)!)^{p^{r-1}}},$$

(8) is equivalent to

$$Q_r \equiv 1 \pmod{p^{r+2}} \qquad (p > 3, r \geqq 1). \tag{9}$$

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A NOTE ON MULTIPLE SERIES OF POSITIVE TERMS

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1. Introduction. It is well known that, if $f(x)$ is a positive function which decreases monotonely towards zero in the interval $0 \leqq x < \infty$, the following theorem holds:

THEOREM 1. *The sequence* $u_n = \sum_{i=0}^{n} f(i) - \int_0^n f(x)dx$ *is convergent and* $\lim u_n = C$, *where* $0 < C < f(0)$.

From the above theorem follow two corollaries:

COROLLARY 1. *The sequence* $u_n(p, q) = \sum_{i=pn}^{qn} f(i) - \int_{pn}^{qn} f(x)dx$ *is convergent to zero if* $0 < p < q$.

COROLLARY 2. *The series* $\sum_{i=0}^{\infty} f(i)$ *converges or diverges with the integral* $\int_0^\infty f(x)dx$ (Maclaurin's integral test).

Theorem 1 and Corollary 1 hold even if $\int_0^\infty f(x)dx$ is divergent. If, in spite of the divergence of the last integral, $\lim_{n\to\infty} \int_{pn}^{qn} f(x)dx = L$ exists, also $\lim_{n\to\infty} \sum_{i=pn}^{qn} f(i) = L$; *e.g.* $\lim_{n\to\infty} \sum_{i=pn}^{qn} 1/i = \log(q/p)$.

If $f(x, y)$ is a positive montonely decreasing function of both variables there are in the literature few generalizations of Corollary 2. Some of them assume that the function $f(x, y)$ is positive outside a certain closed curve in the xy-coordinate plane and diminishes steadily in value as the point (x, y) recedes

from the origin in any direction ([1], p. 359; [2], p. 196). Other authors ([3], p. 86; [4], p. 415) state, in a rather vague way, that the double series converges or diverges with the double integral or discuss only a sufficient condition for convergence ([5], p. 84).

Considering the function $f(x, y) = (x+1)^{-1}e^{-(x+1)y}$ we check that the function is positive and monotonely decreasing in the domain $0 \leqq x < \infty$, $0 \leqq y < \infty$, and that the integral $\int_0^\infty \int_0^\infty (x+1)^{-1}e^{-(x+1)y}dxdy$ exists. However, the double series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)^{-1}e^{-(i+1)j}$$

is divergent because of the divergence of its zero-th row, $\sum_{i=0}^{\infty}(i+1)^{-1}$. This shows that the convergence of the double integral is not a sufficient condition for the convergence of the double series.

In this note we will prove a theorem analogous to Theorem 1 and hence we will develop a necessary and sufficient condition for the convergence of the double series. Then the results will be generalized for functions in any finite number of variables.

2. Integral test for convergence of double series. Denote by $f(x, y)$ a function which has the following properties (A): (A_1) It is integrable and positive in the domain $0 \leqq x < \infty$, $0 \leqq y < \infty$; (A_2) $f(x_1, y_1) \geqq f(x_2, y_2)$ if $x_1 < x_2$, $y_1 < y_2$; (A_3) $\lim f(x, y) = 0$ if $x+y \to \infty$. The properties A_1 and A_2 lead immediately to the inequality

$$(1)\quad \sum_{i=p+1}^{p+n}\sum_{j=q+1}^{q+m} f(i,j) \leqq \int_p^{p+n}\int_q^{q+m} f(x,y)dxdy \leqq \sum_{i=p}^{p+n-1}\sum_{j=q}^{q+m-1} f(i,j), \quad p \geqq 0, q \geqq 0.$$

Since

$$(2)\quad \sum_{i=0}^{n}\sum_{j=0}^{m} f(i,j) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(i,j) + \sum_{i=1}^{n} f(i,0) + \sum_{j=1}^{m} f(0,j) + f(0,0),$$

$$(3)\quad \sum_{i=1}^{n} f(i,0) \leqq \int_0^n f(x,0)dx, \qquad \sum_{j=1}^{m} f(0,j) \leqq \int_0^m f(0,y)dy,$$

the left side of the inequality (1) yields

$$(4)\quad \sum_{i=0}^{n}\sum_{j=0}^{m} f(i,j) - \int_0^n\int_0^m f(x,y)dxdy \leqq \int_0^n f(x,0)dx + \int_0^m f(0,y)dy + f(0,0).$$

Similarly, from the decomposition

$$(5)\quad \sum_{i=0}^{n}\sum_{j=0}^{m} f(i,j) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(i,j) + \sum_{i=0}^{n-1} f(i,m) + \sum_{j=0}^{m-1} f(n,j) + f(n,m),$$

and since

$$(6)\qquad \sum_{i=0}^{n-1} f(i,m) \geqq \int_0^n f(x,m)dx, \qquad \sum_{j=0}^{m-1} f(n,j) \geqq \int_0^m f(n,y)dy,$$

it follows, after application of the right side of the inequality (1), that

$$(7)\qquad 0 < f(n,m) + \int_0^n f(x,m)dx + \int_0^m f(n,y)dy \leqq \sum_{i=0}^{n}\sum_{j=0}^{m} f(i,j) - \int_0^n\int_0^m f(x,y)dxdy.$$

Now, let us assume that $\int_0^\infty f(x,0)dx$ and $\int_0^\infty f(0,y)dy$ are convergent, and let

$$u_n = \sum_{i=0}^{n}\sum_{j=0}^{n} f(i,j) - \int_0^n\int_0^n f(x,y)dxdy.$$

LEMMA. *Let $f(x,y)$ be a function satisfying the conditions (A) and such that $\int_0^\infty f(x,0)dx$ and $\int_0^\infty f(0,y)dy$ exist. Given an $\epsilon>0$, there exists an integer $\mu>0$ such that for $m>\mu$, $\sum_{i=0}^\infty f(i,m)<\epsilon$, and an integer $\nu>0$ such that for $n>\nu$, $\sum_{j=0}^\infty f(n,j)<\epsilon$.*

Since $\int_0^\infty f(x,0)dx$ is convergent so is $\sum_{i=0}^\infty f(i,0)$ and, consequently, $\sum_{i=0}^\infty f(i,m)$. Then, there exists an integer N such that $\sum_{i=N}^\infty f(i,0)<\frac{1}{2}\epsilon$; therefore, also, $\sum_{i=N}^\infty f(i,m)<\frac{1}{2}\epsilon$ for all positive m. Because of property A_3 an integer $\mu>0$ can be found such that for $m>\mu$, $f(0,m)<\frac{1}{2}(\epsilon/N)$. Consequently, $f(i,m)<\frac{1}{2}(\epsilon/N)$ and $\sum_{i=0}^{N-1} f(i,m)<\frac{1}{2}\epsilon$. Finally, it follows that

$$\sum_{i=0}^{\infty} f(i,m) = \sum_{i=0}^{N-1} f(i,m) + \sum_{i=N}^{\infty} f(i,m) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

The second part of the lemma can be proved in similar way.

Now, consider the difference $u_{n+p}-u_n$

$$(8)\qquad \begin{aligned} &= \sum_{i=0}^{n+p}\sum_{j=0}^{n+p} f(i,j) - \int_0^{n+p}\int_0^{n+p} f(x,y)dxdy - \sum_{i=0}^{n}\sum_{j=0}^{n} f(i,j) + \int_0^n\int_0^n f(x,y)dxdy \\ &= \left(\sum_{i=1}^{n}\sum_{j=n+1}^{n+p} f(i,j) - \int_0^n\int_n^{n+p} f(x,y)dxdy\right) \\ &\quad + \left(\sum_{i=n+1}^{n+p}\sum_{j=1}^{n+p} f(i,j) - \int_n^{n+p}\int_0^{n+p} f(x,y)dxdy\right) + \sum_{i=n+1}^{n+p} f(i,0) + \sum_{j=n+1}^{n+p} f(0,j). \end{aligned}$$

Since, according to (1), the expressions in the parentheses are nonpositive, it follows that

$$(9)\qquad u_{n+p} - u_n \leqq \sum_{i=n+1}^{n+p} f(i,0) + \sum_{j=n+1}^{n+p} f(0,j).$$

On the other hand we observe that

$$(10)\qquad u_{n+p}-u_n=\left(\sum_{i=0}^{n}\sum_{j=n}^{n+p}f(i,j)-\int_0^n\int_n^{n+p}f(x,y)dxdy\right)$$
$$+\left(\sum_{i=n}^{n+p}\sum_{j=0}^{n+p}f(i,j)-\int_n^{n+p}\int_0^{n+p}f(x,y)dxdy\right)-\sum_{i=0}^{n}f(i,n)-\sum_{j=0}^{n+p}f(n,j),$$

and, since the quantities in the parentheses are nonnegative, we have

$$(11)\qquad -\left(\sum_{i=0}^{n}f(i,n)+\sum_{j=0}^{n+p}f(n,j)\right)\leqq u_{n+p}-u_n.$$

From (11) and the lemma it follows that there exists an integer $\nu>0$ such that, for $n>\nu$,

$$(12)\qquad -\epsilon\leqq u_{n+p}-u_n.$$

Further, the convergence of $\int_0^\infty f(x,0)dx$ and $\int_0^\infty f(0,y)dy$ assures the existence of an integer $\nu^*>0$ such that, for $n>\nu^*$, the right side of (9) is less than ϵ. Thus we obtain

$$(13)\qquad |u_{n+p}-u_n|<\epsilon,\ n>\max(\nu,\nu^*),\ p \text{ any positive integer},$$

and, hence, a theorem analogous to Theorem 1.

THEOREM 2. *If $f(x,y)$ is a function satisfying the conditions (A) and the integrals $\int_0^\infty f(x,0)dx$ and $\int_0^\infty f(0,y)dy$ are convergent, then the sequence*

$$u_n=\sum_{i=0}^{n}\sum_{j=0}^{n}f(i,j)-\int_0^n\int_0^n f(x,y)dxdy$$

is convergent, and $\lim u_n=C$, *where*

$$0<C<\int_0^\infty f(x,0)dx+\int_0^\infty f(0,y)dy+f(0,0).$$

The following corollary is a consequence of convergence of the sequence u_n.

COROLLARY 3. *If $f(x,y)$ is a function satisfying the conditions (A) and the integrals $\int_0^\infty f(x,0)dx$ and $\int_0^\infty f(0,y)dy$ converge, then the sequence*

$$u_n(p,q)=\sum_{i=pn}^{qn}\sum_{j=pn}^{qn}f(i,j)-\int_{pn}^{qn}\int_{pn}^{qn}f(x,y)dxdy,$$

where $0<p<q$, converges to zero.

The $\lim_{n\to\infty}\int_{pn}^{qn}\int_{pn}^{qn}f(x,y)dxdy$ may exist, although $\int_0^\infty\int_0^\infty f(x,y)dxdy$ does not.

COROLLARY 4. *If* $\lim_{n\to\infty}\int_{pn}^{qn}\int_{pn}^{qn}f(x,y)dxdy=L$, *then* $\lim_{n\to\infty}\sum_{i=pn}^{qn}\sum_{j=pn}^{qn}f(i,j)=L$. For example,

$$\lim_{n\to\infty} \sum_{i=pn}^{qn} \sum_{j=pn}^{qn} (x+y)^{-2} = \log\left[\frac{(p+q)^2}{4pq}\right].$$

Finally, if, also, $\int_0^\infty \int_0^\infty f(x, y)dxdy$ exists, we obtain from Theorem 2 the following criterion for convergence of double series.

THEOREM 3. *The series* $\sum_{i=0}^\infty \sum_{j=0}^\infty f(i, j)$, *where* $f(x, y)$ *is a function possessing the properties* (A), *converges if and only if the integrals* $\int_0^\infty \int_0^\infty f(x, y)dxdy$, $\int_0^\infty f(x, 0)dx$, *and* $\int_0^\infty f(0, y)dy$ *converge.*

3. Generalization. Let $f(x_1, \cdots, x_w)$ be a function which possesses the following properties (B): (B_1) It is positive and integrable in the domain $0 \leqq x_k < \infty$, $k=1, \cdots, w$; (B_2) $f(x_1, \cdots, x_w) \geqq f(y_1, \cdots, y_w)$ if $x_k < y_k$, $k=1, \cdots, w$; (B_3) $\lim f(x_1, \cdots, x_w) = 0$ if $\sum_{k=1}^w x_k \to \infty$.

Further we let:

$$(14)\quad J = J(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_w}^{\beta_w} f(x_1, \cdots, x_w)dx_1 \cdots dx_w,$$

$$(15)\quad S = S(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) = \sum_{i_1=\alpha_1}^{\beta_1} \cdots \sum_{i_w=\alpha_w}^{\beta_w} f(i_1, \cdots, i_w).$$

The properties B_1 and B_2 yield the inequality

$$(16)\quad \begin{aligned} S(\alpha_1 + 1, \beta_1; \cdots; \alpha_w + 1, \beta_w) &\leqq J(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) \\ &\leqq S(\alpha_1, \beta_1 - 1; \cdots; \alpha_w, \beta_w - 1). \end{aligned}$$

If in $f(x_1, \cdots, x_w)$ some $w-r$ variables are replaced by the corresponding lower limits of the integral J and the remaining variables are $x_{k_1}, \cdots, x_{k_r}$, then the function $g(x_{k_1}, \cdots, x_{k_r})$ in r variables is obtained. Similarly, by $G(x_{k_1}, \cdots, x_{k_r})$ we denote the function in r variables if some $w-r$ variables in $f(x_1, \cdots, x_w)$ are replaced by the corresponding upper limits of the integral J. Taking this into account, we set

$$(17)\quad j_r = j_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \int_{\alpha_{k_1}}^{\beta_{k_1}} \cdots \int_{\alpha_{k_r}}^{\beta_{k_r}} g(x_{k_1}, \cdots, x_{k_r})dx_{k_1} \cdots dx_{k_r}$$

$$(18)\quad J_r = J_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \int_{\alpha_{k_1}}^{\beta_{k_1}} \cdots \int_{\alpha_{k_r}}^{\beta_{k_r}} G(x_{k_1}, \cdots, x_{k_r})dx_{k_1} \cdots dx_{k_r}$$

and similarly for the series

$$(19)\quad s_r = s_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \sum_{i_{k_1}=\alpha_{k_1}}^{\beta_{k_1}} \cdots \sum_{i_{k_r}=\alpha_{k_r}}^{\beta_{k_r}} g(i_{k_1}, \cdots, i_{k_r}),$$

$$(20)\quad S_r = S_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \sum_{i_{k_1}=\alpha_{k_1}}^{\beta_{k_1}} \cdots \sum_{i_{k_r}=\alpha_{k_r}}^{\beta_{k_r}} G(i_{k_1}, \cdots, i_{k_r}).$$

There are C_r^w r-tuple integrals (and series) formed in this way. On setting

$$u(n_1, \cdots, n_w; p, q) = S(pn_1, qn_1; \cdots; pn_w, qn_w) - J(pn_1, qn_1; \cdots; pn_w, qn_w) \tag{21}$$

and because of the identities

$$\begin{aligned} S(pn_1, qn_1; \cdots; pn_w, qn_w) &= S(pn_1 + 1, qn_1; \cdots; pn_w + 1, qn_w) \\ &\quad + \sum s_{w-1} + \cdots + \sum s_1 + f(pn_1, \cdots, pn_w) \\ &= S(pn_1, qn_1 - 1; \cdots; pn_w, qn_w - 1) \\ &\quad + \sum S_{w-1} + \cdots + \sum S_1 + f(qn_1, \cdots, qn_w), \end{aligned} \tag{22}$$

where each summation is taken over C_r^w r-tuple series ($r = 1, \cdots, w-1$), we obtain from (16),

$$\begin{aligned} 0 &< f(qn_1, \cdots, qn_w) + \sum J_1 + \cdots + \sum J_{w-1} \\ &\leqq u(n_1, \cdots, n_w; p, q) \leqq f(pn_1, \cdots, pn_w) + \sum j_1 + \cdots + \sum j_{w-1}. \end{aligned} \tag{23}$$

Since the lemma can also be generalized, we obtain from (23) a theorem analogous to Theorem 1:

THEOREM 4. *If all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r = 1, \cdots, w-1$, *are convergent, then* $\lim u(n, \cdots, n; 0, 1) = C$, *where*

$$0 < C < \sum j_1 + \cdots + \sum j_{w-1} + f(0, \cdots, 0).$$

Hence we have the following corollaries:

COROLLARY 5. *If* $0 < p < q$ *and all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r = 1, \cdots, w-1$, *are convergent, then* $\lim u(n, \cdots, n; p, q) = 0$.

COROLLARY 6. *If* $\lim J(pn, qn; \cdots; pn, qn) = L$ *then also*

$$\lim S(pn, qn; \cdots; pn, qn) = L.$$

Finally, if the integral $J(0, \infty; \cdots; 0, \infty)$ also converges we obtain

THEOREM 5. *The series* $S(0, \infty; \cdots; 0, \infty)$ *is convergent if and only if the integral* $J(0, \infty; \cdots; 0, \infty)$ *and all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r = 1, \cdots, w-1$, *converge.*

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A CHARACTERIZATION OF PSEUDO-CHAIN MAPPINGS IN MAYER COMPLEXES

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1. Introduction. Let $K=\{C_p(K), \partial_p^K\}$ and $L=\{C_p(L), \partial_p^L\}$ be two Mayer complexes† and for each integer p, $f_p: C_p(K)\to C_p(L)$ a homomorphism. Such a system of homomorphisms is called a mapping from K to L and is denoted by $f: K\to L$. A mapping $f: K\to L$ is called a chain-mapping if and only if $f_{p-1}\partial_p^K = \partial_p^L f_p$ for every integer p and such mappings have been studied extensively. Their main virtue is that cycles are transformed into cycles and boundaries are transformed into boundaries. A chain mapping $f: K\to L$ then induces in a natural way a homomorphism $f_p^*: H_p(K)\to H_p(L)$ of the p-homology groups of K and L respectively.

2. Pseudo-chain mappings. Let $f: K\to L$ be a mapping from K to L, *i.e.*, for each integer p, $f_p: C_p(K)\to C_p(L)$ is a homomorphism. Let $Z_p(K)$ and $Z_p(L)$ be the groups of p-cycles in $C_p(K)$ and $C_p(L)$, respectively, and $B_p(K)$ and $B_p(L)$ the groups of p-boundaries in $C_p(K)$ and $C_p(L)$ respectively.

DEFINITION 1. *A mapping $f: K\to L$ will be termed a pseudo-chain mapping if and only if for each integer p,* (1) $f_p\{Z_p(K)\}\subset Z_p(L)$ *and* (2) $f_p\{B_p(K)\}\subset B_p(L)$.

In Section 4, we will obtain a characterization of such mappings in terms of a homomorphism from $C_p(K)$ to $C_{p-1}(L)$ to be introduced in Section 3.

Example 1. A pseudo-chain mapping is not necessarily a chain-mapping. For let K and L be abstract simplicial complexes consisting of simplexes (ab), (a), (b) and (xy), (x), (y), respectively. Suppose G is any abelian group. $C_p(K, G)$ and $C_p(L, G)$ will denote the group of p-chains of K over G and L over G, respectively. Define $f_1: C_1(K, G)\to C_1(L, G)$ as follows: $f_1\{g[ab]\}=g[xy]$, where $[ab]$ and $[xy]$ denote oriented 1-simplexes. Define $f_0: C_0(K, G)\to C_0(L, G)$ as follows: $f_0\{g_1[a]+g_2[b]\}=g_1[y]+g_2[x]$. The reader can easily verify that $f: K\to L$ is a pseudo-chain mapping. But it is not a chain mapping, for

$$\partial_1^L f_1 g[ab] = \partial_1^L g[xy] = g[y] - g[x],$$
$$f_0\partial_1^K g[ab] = f_0\{g[b] - g[a]\} = g[x] - g[y].$$

Example 2. Let K and L be abstract simplicial complexes consisting of simplexes (abc), (ab), (bc), (ca), (a), (b), (c) and (xy), (yz), (zx), (x), (y), (z), respectively. Let G be any abelian group. Define $f_2: C_2(K, G)\to G_2(L, G)$ to be trivial and $f_1: C_1(K, G)\to C_1(L, G)$ as follows:

$$f_1\{g_1[ab] + g_2[bc] + g_3[ca]\} = g_1[xy] + g_2[yz] + g_3[zx]$$

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† For every integer p let C_p be an additive abelian group and $\partial_p: C_p\to C_{p-1}$ a homomorphism satisfying the condition $\partial_{p-1}\partial_p\equiv 0$, i.e., for $c_p\in C_p$, then $\partial_{p-1}\partial_p c_p$ is the identity of C_{p-2}. A collection $\{C_p, \partial_p\}$ satisfying these conditions is called a Mayer Complex.

and f_0: $C_0(K, G) \to C_0(L, G)$ as follows:

$$f_0\{g_1[a] + g_2[b] + g_3[c]\} = g_1[x] + g_2[y] + g_3[z].$$

The reader will easily verify that f: $K \to L$ is a mapping, that $f_p\{Z_p(K, G)\} \subset Z_p(L, G)$ for all p, but $f_1\{B_1(K, G)\} \not\subset B_1(L, G)$.

Example 3. Let K and L be abstract simplicial complexes consisting of simplexes (ab), (bc), (ca), (a), (b), (c) and (xy), (yz), (x), (y), (z) respectively. G will be the coefficient group. Define f: $K \to L$ as follows:

$$f_1\{g_1[ab] + g_2[bc] + g_3[ca]\} = g_1[xy] + g_2[yz],$$
$$f_0\{g_1[a] + g_2[b] + g_3[c]\} = 0.$$

The reader can verify that f: $K \to L$ is a mapping, and that $f_p\{B_p(K, G)\} \subset B_p(L, G)$ for all p, but $f_1\{Z_1(K, G)\} \not\subset Z_1(L, G)$.

Thus Examples 2 and 3 show that the conditions (1) $f_p\{Z_p(K, G)\} \subset Z_p(L, G)$ and (2) $f_p\{B_p(K, G)\} \subset B_p(L, G)$ are independent.

3. The Δ-function. Let $K = \{C_p(K), \partial_p^K\}$ and $L = \{C_p(L), \partial_p^L\}$ be Mayer complexes and f: $K \to L$ a mapping, *i.e.*, for each p, f_p: $C_p(K) \to C_p(L)$ is a homomorphism.

DEFINITION 2. $\Delta(f_p)$: $C_p(K) \to C_{p-1}(L)$ as follows: $\Delta(f_p) = f_{p-1}\partial_p^K - \partial_p^L f_p$ for all integers p.

$\Delta(f_p)$ is clearly a homomorphism from $C_p(K)$ to $C_{p-1}(L)$.

THEOREM 1. *Let K and L be Mayer complexes and f: $K \to L$ a mapping. Then* $\Delta(f_p)\{Z_p(K)\} \subset B_{p-1}(L)$.

Proof. Let $z_p^K \in Z_p(K)$. Then

$$\Delta(f_p)\, z_p^K = (f_{p-1}\partial_p^K - \partial_p^L f_p) z_p^K = f_{p-1}\partial_p^K z_p^K - \partial_p^L f_p z_p^K$$
$$= f_{p-1}0 - \partial_p^L f_p z_p^K = \partial_p^L f_p(-z_p^K) \in B_{p-1}(L).$$

COROLLARY 1. *Let K and L be Mayer complexes and f: $K \to L$ a mapping. Then $\Delta(f_p)$ induces the trivial homomorphism $\Delta(f_p)_*$: $H_p(K) \to H_{p-1}(L)$ on the homology groups of K and L respectively.*

Proof. Let $[z_p^K] \in H_p(K)$, *i.e.*, $[z_p^K]$ is the class of p-cycles homologous to z_p^K. Then $\Delta(f_p)_*\{[z_p^K]\} = [\Delta(f_p) z_p^K] = [b_{p-1}^L]$ (by Th. 1) $= [0]$.

THEOREM 2. *Let K, L, M be Mayer complexes and f: $K \to L$ and g: $L \to M$ be mappings. Then $\Delta(g_p f_p) = g_{p-1}\Delta(f_p) + \Delta(g_p) f_p$.*

Proof.

$$\Delta(g_p f_p) = g_{p-1} f_{p-1}\partial_p^K - \partial_p^M g_p f_p$$

$$= g_{p-1}f_{p-1}\partial_p^K - g_{p-1}\partial_p^L f_p + g_{p-1}\partial_p^L f_p - \partial_p^M g_p f_p$$
$$= g_{p-1}\{f_{p-1}\partial_p^K - \partial_p^L f_p\} + \{g_{p-1}\partial_p^L - \partial_p^M g_p\}f_p$$
$$= g_{p-1}\Delta(f_p) + \Delta(g_p)f_p.$$

THEOREM 3. *Let K and L be Mayer complexes and $f: K \to L$ a mapping. Then* $\partial_{p-1}^L \Delta(f_p) + \Delta(f_{p-1})\partial_p^K = 0$.

Proof.

$$\partial_{p-1}^L \Delta(f_p) + \Delta(f_{p-1})\partial_p^K = \partial_{p-1}^L\{f_{p-1}\partial_p^K - \partial_p^L f_p\} + \{f_{p-2}\partial_{p-1}^K - \partial_{p-1}^L f_{p-1}\}\partial_p^K$$
$$= \partial_{p-1}^L f_{p-1}\partial_p^K - \partial_{p-1}^L f_{p-1}\partial_p^K = 0.$$

THEOREM 4. *Let K, L, M be Mayer complexes and $f: K \to L$ and $g: L \to M$ be mappings. Then* $\Delta(g_{p-1})\Delta(f_p) + \partial_{p-1}^M \Delta(g_p f_p) = g_{p-2}\partial_{p-1}^L f_{p-1}\partial_p^K + \partial_{p-1}^M g_{p-1}\partial_p^L f_p$.

Proof.

$$\Delta(g_{p-1})\Delta(f_p) + \partial_{p-1}^M \Delta(g_p f_p)$$
$$= (g_{p-2}\partial_{p-1}^L - \partial_{p-1}^M g_{p-1})(f_{p-1}\partial_p^K - \partial_p^L f_p) + \partial_{p-1}^M\{g_{p-1}f_{p-1}\partial_p^K - \partial_p^M g_p f_p\}$$
$$= g_{p-2}\partial_{p-1}^L f_{p-1}\partial_p^K - \partial_{p-1}^K g_{p-1}f_{p-1}\partial_p^K + \partial_{p-1}^M g_{p-1}\partial_p^L f_p + \partial_{p-1}^M g_{p-1}f_{p-1}\partial_p^K$$
$$= g_{p-2}\partial_{p-1}^L f_{p-1}\partial_p^K + \partial_{p-1}^M g_{p-1}\partial_p^L f_p.$$

THEOREM 5. *Let K, L, M be Mayer complexes and $f: K \to L$ and $g: L \to M$ be mappings. Then* $g_{p-2}\Delta(f_{p-1})\partial_p^K + \partial_{p-1}^M \Delta(g_p f_p) + \Delta(g_{p-1})\Delta(f_p) = \partial_{p-1}^M \Delta(g_p)f_p$.

Proof. By Theorem 4 it suffices to show that

$$g_{p-2}\Delta(f_{p-1})\partial_p^K - \partial_{p-1}^M \Delta(g_p)f_p = -g_{p-2}\partial_{p-1}^L f_{p-1}\partial_p^K - \partial_{p-1}^M g_{p-1}\partial_p^L f_p.$$

This easy verification is left to the reader.

4. A characterization of pseudo-chain mappings.

LEMMA 1. *Let $f: K \to L$ be a mapping, where K and L are Mayer complexes. Then $f_p\{Z_p(K)\} \subset Z_p(L)$ if and only if $\Delta(f_p)\{Z_p(K)\} = 0$.*

Proof. Sufficiency. Let $\Delta(f_p)\{Z_p(K)\} = 0$. Take $z_p^K \in Z_p(K)$ and we assert that $f_p z_p^K \in Z_p(L)$. Now

$$\partial_p^L f_p z_p^K = f_{p-1}\partial_p^K z_p^K - \Delta(f_p)z_p^K = f_{p-1}0 - \Delta(f_p)z_p^K = 0.$$

Thus $f_p z_p^K \in Z_p(L)$.

Necessity. Let $z_p^K \in Z_p(K)$. Assert $\Delta(f_p)z_p^K = 0$. Now

$$\Delta(f_p)z_p^K = (f_{p-1}\partial_p^K - \partial_p^K f_p)z_p^K = f_{p-1}\partial_p^K z_p^K - \partial_p^K f_p z_p^K = 0.$$

LEMMA 2. *Let $f: K \to L$ be a mapping. Then $f_p\{B_p(K)\} \subset B_p(L)$ if and only if $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$.*

Proof. Sufficiency. Suppose $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$ and take $b_p^K \in B_p(K)$. Then $f_p b_p^K = f_p \partial_{p+1}^K c_{p+1}^K = \Delta(f_{p+1}) c_{p+1}^K + \partial_{p+1}^L f_{p+1} c_{p+1}^K$. Thus $f_p b_p^K \in B_p(L)$.

Necessity. Let $f_p\{B_p(K)\} \subset B_p(L)$ and take $c_p^K \in C_p(K)$. Then $\Delta(f_p)c_p^K = (f_{p-1}\partial_p^K - \partial_p^L f_p)c_p^K = f_{p-1}\partial_p^K c_p^K - \partial_p^L f_p c_p^K$. But $f_{p-1}\partial_p^K c_p^K \in B_{p-1}(L)$ and hence $\Delta(f_p)c_p^K \in B_{p-1}(L)$.

THEOREM 6. *Let $f: K \to L$ be a mapping. Then $f: K \to L$ is a pseudo-chain mapping if and only if* (1) $\Delta(f_p)\{Z_p(K)\} = 0$ and (2) $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$.

The proof follows immediately from the above two lemmas.

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LINEAR TRANSFORMATIONS OF SEQUENCES

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Given a sequence $\{s_m\}$ $(m = 1, 2, \cdots)$ of real numbers we define s_m' for $m \geqq n$ by $s_m' = \alpha_1 s_m + \cdots + \alpha_n s_{m-n+1}$ where the α_i are real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$.

Kubota [3] has shown that the convergence of $\{s_m'\}$ to a finite limit s implies the convergence of $\{s_m\}$ to s if and only if $f(z) \equiv \alpha_1 z^{n-1} + \cdots + \alpha_n$ has no zeros in $|z| \geqq 1$. Conditions for this have recently been investigated in [1] and [2]. A simple sufficiency condition will here be deduced from the following result obtained by Parodi [4] by an easy application of Rouché's theorem on the zeros of a function:

If $\lambda_1, \cdots, \lambda_n$ are positive numbers such that $\lambda_1 + \cdots + \lambda_n = 1$ and $M = \max_{i=1,\cdots,n} |a_i/\lambda_i|^{1/i}$ then all the roots of $z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ lie in $|z| \leqq M$.

THEOREM. *If $\alpha_1 s_m + \cdots + \alpha_n s_{m-n+1}$ tends to a finite limit s as m tends to infinity, where the α_i are real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$ and $|\alpha_1| > |\alpha_2| + \cdots + |\alpha_n|$, then s_m tends to s as m tends to infinity.*

Proof. Clearly $\alpha_1 \neq 0$ so that the numbers $a_i = \alpha_{i+1}/\alpha_1$, $(i = 1 \text{ to } n-1)$, are finite. Also, if we put $|\alpha_1| - |\alpha_2| - \cdots - |\alpha_n| = \epsilon(n-1)|\alpha_1|$, then $\epsilon > 0$. Next, if we put $\lambda_i = |a_i| + \epsilon$, $(i = 1 \text{ to } n-1)$, then the λ_i are all positive, and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} = (|\alpha_2| + \cdots + |\alpha_n|)/|\alpha_1| + (n-1)\epsilon = 1.$$

On applying Parodi's theorem with n replaced by $n-1$ we have

$$M = \max_{i=1,\cdots,n-1} \{|a_i|/(|a_i| + \epsilon)\}^{1/i} < 1,$$

so that all the roots of

$$\frac{f(z)}{\alpha_1} \equiv z^{n-1} + \frac{\alpha_2}{\alpha_1} z^{n-2} + \cdots + \frac{\alpha_{n-1}}{\alpha_1} z + \frac{\alpha_n}{\alpha_1} = 0$$

lie in $|z| \leqq M < 1$. It follows from Kubota's result stated above that if $\{s'_m\}$ converges to a finite limit then $\{s_m\}$ converges to the same limit.

Remarks. Dr. D. Borwein has drawn my attention to the fact that as a by-product of his work on power series, Schur [5; Satz 17] found that the zeros of the $(n-1)$th degree polynomial $c_1 z^{n-1} + \cdots + c_{n-1} z + c_n$ lie in $|z| < 1$ if and only if all the determinants

$$\Delta_{r+1} = \begin{vmatrix} c_1 & 0 & \cdots & 0 & c_n & c_{n-1} & \cdots & c_{n-r} \\ c_2 & c_1 & \cdots & 0 & 0 & c_n & \cdots & c_{n-r+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{r+1} & c_r & \cdots & c_1 & 0 & 0 & \cdots & c_n \\ c_n & 0 & \cdots & 0 & c_1 & c_2 & \cdots & c_{r+1} \\ c_{n-1} & c_n & \cdots & 0 & 0 & c_1 & \cdots & c_r \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{n-r} & c_{n-r+1} & \cdots & c_n & 0 & 0 & \cdots & c_1 \end{vmatrix}$$

are strictly positive for $r = 0$ to $n-2$. By Kubota's result this therefore provides a set of necessary and sufficient conditions for the convergence of $\{s'_m\}$ to a finite limit to imply the convergence of $\{s_m\}$ to the same limit. When $n = 3$ and $\alpha_1 \neq 0$ these conditions reduce, as do those found in [1], to $\alpha_2 < \frac{1}{2}$ and $2\alpha_1 + \alpha_2 > 1$. In this case the condition $|\alpha_1| > |\alpha_2| + |\alpha_3| = |\alpha_2| + |1 - \alpha_1 - \alpha_2|$ is satisfied if and only if $\alpha_1 > \frac{1}{2}$, $\alpha_2 < \frac{1}{2}$ and $\alpha_1 + \alpha_2 > 1$. The condition derived here is therefore a sufficient but not a necessary one. For $n \geqq 4$, however, it is much easier to apply than Schur's conditions or than those derived by the method indicated in [2].

The simplicity of the above condition also suggests that it should be possible to find a proof of the theorem depending entirely on real-variable arguments.

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MARKOV CHAINS AS RANDOM INPUT AUTOMATA*

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By making use of certain matrices which have direct probabilistic interpretations (thus avoiding eigenvalues), Kemeny and Snell [1] have recently given a unified and very meaningful presentation of the theory of finite Markov chains. In the present note I wish to indicate a somewhat different matrix-theoretic approach. Its significant feature is that it establishes a connection between Markov chains and automata theory. The connection is based on a theorem which is reminiscent of Birkhoff's result concerning doubly stochastic matrices [2] and which leads to a simple proof of the fundamental theorem (generalized) of regular finite Markov chains.

A *finite automaton* (or *finite deterministic system with input*) consists of a set X of *states* together with a set $\{\Gamma_\theta\}$ of mappings $\Gamma_\theta: X \to X$ which determine the behavior of the system for each value of the *input parameter* θ. We shall call such a system $(X, \{\Gamma_\theta\})$ a *box* for short. (This is suggested by E. F. Moore's use of the term "black box" [3], viz., a box plus a function which assigns to each state a value from some set of "outputs.") A box of order n (*i.e.*, with n states) can have at most n^n different input values. There exist exactly $2^{n^n}-1$ different boxes of order n. (Thus we require that neither X nor $\{\Gamma_\theta\}$ be void.)

We think of a box as moving through its states in discrete units of time. Given an initial state and a sequence of input values, the sequence of states through which a box moves is completely determined. An elevator in an n-story building is a box whose states are the different floors and whose input is provided by the call-buttons on each floor and the control buttons within. These ideas are more fully discussed in [4].

A *finite Markov chain* consists of a finite set X of states together with a probability rule (p_{ij}) which specifies, for each pair $x_i, x_j \in X$, the probability that the system moves to state x_j, given that it is in state x_i. The system $(X, (p_{ij}))$ will be called a *chain* for short. Given an initial state, the behavior of a chain is completely (but stochastically) determined by the p_{ij}. The reader is referred to [5] for examples and for a brief survey of the Kemeny and Snell approach.

Consider now a box whose consecutive input values are selected randomly and independently by a sampling device of some kind. The box $(X, \{\Gamma_\theta\})$ together with the probability distribution $q(\theta)$ over the input values will be called a *box with random input*. Its behavior is exactly that of a Markov chain. More remarkable, however, is that the converse is also true.

Every box with random input realizes a chain of the same order. Conversely, for every chain of order n, a box of order n with random input can be constructed as a realization. For this construction, no more than n^2-n+1 *input values need be used.*

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* This research is supported by a National Science Foundation Cooperative Fellowship.

Actually, this is a corollary to a more general result which in a moment we shall state and prove (in slightly different language). Here we note a possible application. Suppose an investigator has before him a rather complex system which he discovers to be a finite Markov chain. He suspects that the system is actually deterministic (in some relatively simple way) and that its probabilistic behavior is due to external sources of random influence of which he may or may not be aware. The above theorem says that his conjecture is not at all a logical impossibility and that the number of input values may be reasonably few. (In any case, there can be no more than $n \log_2 n$ bits of imput variation.)

At this point it is convenient to use matrix terminology. Just as with each chain we associate an $n \times n$ matrix (p_{ij}), with each box we can associate a set of $n \times n$ matrices D_θ, one for each mapping Γ_θ, in the following manner: If θ is held fixed, the mapping $\Gamma_\theta: X \to X$ describes a chain in which all probabilities are 0 or 1. D_θ is just the corresponding $n \times n$ matrix of (degenerate) probabilities. But now we wish to generalize to rectangular matrices.

An $m \times n$ matrix (p_{ij}) is *stochastic* if all entries are nonnegative and each row sums to 1. Hence the totality of all $m \times n$ stochastic matrices forms a convex set of dimension $m(n-1)$. Contained in this set are the $m \times n$ *deterministic matrices* of the form $D(j_1, \cdots, j_m)$, wherein the ith row has 1 in column j_i and zeros elsewhere. A deterministic matrix of the form $D(k, \cdots, k)$ will be called a *star*.

THEOREM 1. *An $m \times n$ matrix M is stochastic if and only if it is a barycenter of $m \times n$ deterministic matrices*:

$$M = q_1 D_1 + \cdots + q_r D_r \left(\sum_i q_i = 1,\ q_i \geqq 0 \right).$$

In this expression, r need be no greater than $m(n-1)+1$.

Remark. By specializing to square matrices and reinterpreting in terms of chains and boxes with random input (given by the distribution $q_1, \cdots, q_r$), we obtain the formulation which originally suggested this theorem.

Proof. That a barycenter is stochastic follows immediately from the fact that we are working in a convex set which contains the D_i. Suppose now that $M = (p_{ij})$ is stochastic. We assert that

$$M = \sum_{j_1 \cdots j_m} p_{1j_1} \cdots p_{mj_m} D(j_1, \cdots, j_m), \tag{1}$$

where the j_i run independently from 1 to n, and that the sum of the coefficients is 1. To obtain the entry in the ath row and bth column, we add up the coefficients of the matrices $D(j_1, \cdots, j_{a-1}, b, j_{a+1}, \cdots, j_m)$ over all j_i $(i \neq a)$. Since for each i, $\sum_{j_i} p_{ij_i} = 1$, we get precisely p_{ab}, as desired. For the same reason, the sum of all coefficients is 1, so that M is indeed a barycenter of deterministic matrices. That their number may be reduced to $m(n-1)+1$ follows from the fact that the

convex set is of dimension $m(n-1)$.

Equation (1) will be referred to as *the representation of M in expanded form.* Let the terms be numbered so that the stars come first, *i.e.*, so that $D_k = D(k, \cdots, k)$ for $k=1, \cdots, n$.

Now specialize once more to square matrices. We note that under the rule for multiplying $n \times n$ deterministic matrices,

$$D(a_1, \cdots, a_n) D(b_1, \cdots, b_n) = D(b_{a_1}, \cdots, b_{a_n}),$$

stars are preserved: $D(a_1, \cdots, a_n) D_k = D_k$ and $D_k D(b_1, \cdots, b_n) = D_{b_k}$, for $k=1, \cdots, n$. This fact, together with the representation of a stochastic matrix in expanded form, gives us

THEOREM 2. *Let a square stochastic matrix M be called* ***semiregular*** *if for some t a star appears in the representation of M^t in expanded form. Let the* ***degree of regularity*** *be the maximum number of stars that appear (for any power t). Then, if M is semiregular, its powers converge to a matrix whose rows are all identical and whose degree of regularity is the same as that of M.*

Proof. Let S_t be the sum of the coefficients of the stars in $M^t = q_1^{(t)} D_1 + \cdots + q_{n^n}^{(t)} D_{n^n}$. Since M is semiregular, $S_t \neq 0$ for some t. We might as well assume $S_1 \neq 0$. We shall show that $\lim_{t \to \infty} S_t = 1$, thus that $\lim_{t \to \infty} M^t = q_1^* D_1 + \cdots + q_n^* D_n$ ($\sum_i q_i^* = 1$). That the $q_i^{(t)}$ form nondecreasing sequences bounded above (by 1), and hence that these converge to the q_i^* for $1 \leqq i \leqq n$, will be evident from the way stars are preserved under multiplication. Consider $M^{t+1} = MM^t$:

$$\left[\sum_{i=1}^{n^n} q_i D_i \right] \left[\sum_{j=1}^{n^n} q_j^{(t)} D_j \right]$$

$$= \sum_{i=1}^{n^n} q_i D_i \sum_{j=1}^{n^n} q_j^{(t)} D_j + \sum_{i=1}^{n^n} q_i D_i \sum_{j=n+1}^{n^n} q_j^{(t)} D_j + \text{other terms}$$

$$= \sum_{j=1}^{n} q_j^{(t)} D_j + \sum_{i=1}^{n} q_i \sum_{j=n+1}^{n^n} q_j^{(t)} D_{i(j)} + \text{other terms},$$

where $1 \leqq i(j) \leqq n$, by the product rule for stars. Collecting the coefficients of the stars in sight, we get $S_{t+1} \geqq S_t + S_1(1 - S_t)$. Then $1 - S_{t+1} \leqq (1 - S_1)(1 - S_t) \leqq (1 - S_1)^{t+1}$. Thus if $S_1 \neq 0$, then $1 - S_t \to 0$, as asserted.

Note that a semiregular matrix can be recognized by the fact that one of its powers has a column with nonvanishing entries. Semiregular matrices are not associated with ergodic chains unless the degree of regularity is n, in which case Theorem 2 is just the fundamental theorem for regular chains.

Addendum: Within the context of the theory of convex polyhedra, Birkhoff's theorem is also an easy consequence of Theorem 1. By definition, the set of all $n \times n$ doubly stochastic matrices is the intersection of the set of $n \times n$ stochastic matrices with the set of their transposes. By Theorem 1 (and its "transpose"),

this is the intersection of two $n(n-1)$-dimensional convex polyhedra in n^2-dimensional space, hence is itself a convex polyhedron (of dimension $(n-1)^2$). The vertices are just the $n \times n$ permutation matrices. For if M is doubly stochastic, but not a permutation matrix, then M contains at least four elements p_{ab}, p_{ac}, p_{db}, p_{dc}, such that $\min(p_{ab}, 1-p_{ab}, \cdots, 1-p_{dc}) = \epsilon > 0$. Construct M_1 and M_2 by replacing these four elements in M with $p_{ab}+\epsilon$, $p_{ac}-\epsilon$, $p_{db}-\epsilon$, $p_{dc}+\epsilon$ and $p_{ab}-\epsilon$, $p_{ac}+\epsilon$, $p_{db}+\epsilon$, $p_{dc}-\epsilon$, respectively. Then $M = \frac{1}{2}M_1 + \frac{1}{2}M_2$ lies between two doubly stochastic matrices and so is not a vertex. In short, the set of $n \times n$ doubly stochastic matrices is a convex polyhedron whose vertices are the $n \times n$ permutation matrices. This is Birkhoff's theorem.

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CORRECTION

A paper by S. Chowla and F. B. Correia (this MONTHLY, vol. 67, 1960, p. 884) pointed out the falsity of the conjecture $p_1 \cdot p_2 \cdot \cdots \cdot p_{\pi(N)} \sim e^N$ made by E. Trost in his book *Primzahlen, Bd. II* (Basel-Stuttgart, 1953). Professor Lowell Schoenfeld has pointed out that this error was noted by him in his review of this book (Bull. Amer. Math. Soc., vol. 62, 1956, pp. 54–57).

CLASSROOM NOTES

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A PROOF OF THE EULER-FERMAT THEOREM

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1. Introduction. The object here is to present a proof of the Euler-Fermat theorem—namely, *if* $(x, m) = 1$, *then* $x^{\phi(m)} - 1 \equiv 0 \pmod{m}$—that is based upon (i) the Fermat theorem (special case of the foregoing when m is prime) and (ii) a standard method for determining solutions of a polynomial congruence $f(x) \equiv 0 \pmod{p^k}$ from knowledge of the solutions of $f(x) \equiv 0 \pmod{p^{k-1}}$. Here, as usual, $\phi(m)$ is the number of positive integers $\leq m$ and relatively prime to m.

Throughout we set

$$f(x) = x^{p-1} - 1, \qquad f'(x) = (p-1)x^{p-2}, \tag{1}$$

the latter being the derivative of the former; p denotes an arbitrary prime. We may thus state (and we accept as given) the Fermat theorem in the form

$$f(x) \equiv 0 \pmod{p} \text{ provided that } (x, p) = 1. \tag{2}$$

The standard method (ii) alluded to above reads as follows:

If y is any solution of $f(x) \equiv 0 \pmod{p^{k-1}}$, then

$$z = y + tp^{k-1}, \quad \textit{where} \quad f'(y)t \equiv -\left[f(y)/p^{k-1}\right] \pmod{p}, \tag{3}$$

satisfies $f(z) \equiv 0 \pmod{p^k}$, provided that $f'(y) \not\equiv 0 \pmod{p}$. (See [1], pp. 96–97.)

2. Proof for $m = p^r$. Since $\phi(p^r) = p^{r-1}(p-1)$, the Euler-Fermat theorem for $m = p^r$ assumes the form $f(x^{p^{r-1}}) \equiv 0 \pmod{p^r}$ for all x satisfying $(x, p) = 1$. By the Fermat theorem (2) this holds for $r = 1$; we suppose that it holds for $r = k-1$ and proceed to demonstrate that it then holds for $r = k$ $(k \geqq 2)$. By the mathematical-induction principle, its validity for all positive integers r then follows.

With u arbitrary to within $(u, p) = 1$, our assumption is that $y = u^{p^{k-2}}$ is a solution of $f(x) \equiv 0 \pmod{p^{k-1}}$. According to (1), the second of (3) reads

$$(p-1)y^{p-2}t \equiv -\left[(y^{p-1} - 1)/p^{k-1}\right] \pmod{p} \tag{4}$$

—with $(u, p) = 1$, and consequently $(y, p) = 1$, insuring that $f'(y) \not\equiv 0 \pmod{p}$. Since $(p-1) \equiv -1 \pmod{p}$ and $y^{p-1} \equiv 1 \pmod{p}$, by (2), multiplication of both sides of (4) by $-y$ yields $t \equiv \left[(y^p - y)/p^{k-1}\right] \pmod{p}$. Thus, according to (3), we have that $z = y + \left[(y^p - y)/p^{k-1}\right]p^{k-1} = y^p = u^{p^{k-1}}$ satisfies $f(z) \equiv 0 \pmod{p^k}$; that is, $f(u^{p^{k-1}}) \equiv 0 \pmod{p^k}$, which was to have been demonstrated.

3. Proof for arbitrary m. Passage from the case $m = p^r$ to general $m = \prod_{j=1}^{n} p_j^{r_j}$, where $p_1, \cdots, p_n$ are distinct primes, is effected through (i) $\phi(m) = \prod_{j=1}^{n} \phi(p_j^{r_j})$ and (ii) if $a \equiv b \pmod{p_j^{r_j}}$ for all $j = 1, \cdots, n$, then $a \equiv b \pmod{m}$. Indeed, if ϕ_j denotes $\phi(p_j^{r_j})$, then $x^{\phi_j} \equiv 1 \pmod{p_j^{r_j}}$ $j = 1, \cdots, n$, provided that $(x, m) = 1$, by Section 2. We proceed, for each $j = 1, \cdots, n$, to raise the jth of these n congruences to the power $\left[\phi(m)/\phi(p_j^{r_j})\right]$, from which we obtain

$$x^{\phi(m)} \equiv 1 \pmod{p_j^{r_j}}.$$

By (ii) we therefore have $x^{\phi(m)} \equiv 1 \pmod{m}$ whenever $(x, m) = 1$, which is the Euler-Fermat theorem.

4. Remark. The result achieved in Section 2 is closely related to the theorem, proved in [1], page 97, to the effect that $x^{p-1} - 1 \equiv 0 \pmod{p^r}$ has precisely $p-1$ roots for each positive integer r. These roots are identified in Section 2 as $u^{p^{r-1}}$ for $u = 1, \cdots, p-1$.

Reference

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THE GENERAL MONGÉ EQUATION AND ITS EXTENSION

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In the general nonlinear partial differential equation of Mongé,

$$Rr + Ss + Tt + U(rt - s^2) = V$$

$$(1) \qquad p \equiv \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2},$$

it will be assumed here that R, S, T, U, and V are functions of x, y, z, p, and q. By making use of the identities,

$$(2) \qquad \begin{aligned} &pdx + qdy - dz = 0, \quad && (rt - s^2)dx + sdq - tdp = 0, \\ &rdx + sdy - dp = 0, \quad && (rt - s^2)dy + sdp - rdq = 0, \\ &sdx + tdy - dq = 0, \end{aligned}$$

and combinations of these, an equation of the Mongé type is derived; and, in terms of quantities used, total differential equations are established for finding whatever integrals may exist. These integrals then are given by (7) and (8). Although these total differential equations can be derived from the pair ordinarily found in textbooks on the subject, they have the advantage also that the usual test for integrability may be applied. Still another advantage inherent in the method offered here is that it may be extended to equations involving more than the two independent variables x and y appearing in (1).

Although the right pair of (2) is obtained from the second and third of the left members—and the equations are therefore not independent—it will be seen that certain advantages are gained by including them in arriving at (1) and intermediate integrals. The new approach offered here consists in introducing auxiliary functions of x, y, z, p, and q—namely, P, Q, X, and Y, where $PQXY \neq 0$. Then, after (2), excluding the first in the left members, are multiplied, respectively, by $RSVY$, $STVX$, $QSTU$, $PRSU$, and added, the results may be put in the form,

$$(3) \qquad \begin{aligned} &S(VYdx - PUdq)Rr + [U(PRdp + QTdq) + V(TXdx + RYdy)]Ss \\ &+ S(VXdy - QUdp)Tt + S(QTdx + PRdy)U(rt - s^2) = S(RYdp + TXdq)V. \end{aligned}$$

Now, this equation is equivalent to equation (1), provided that $S \neq 0$ was assumed and

$$(4) \qquad \begin{aligned} VYdx - PUdq &= VXdy - QUdp = QTdx + PRdy = RYdp + TXdq \\ &= S^{-1}[U(PRdp + QTdq) + V(TXdx + RYdy)]. \end{aligned}$$

In (4), it is possible to define Q_1, Q_2, W_1, and W_2 in such a way that

$$(5) \qquad \begin{aligned} QT - VY = PQ_1, \quad PU + TX = -Q_2Y, \quad PR - VX = QW_2, \\ QU + RY = -W_1X. \end{aligned}$$

Recalling now that, by hypothesis, $PQXY \neq 0$, one may conclude that

$$(6)\qquad \begin{aligned} &Q_1dx + Rdy + Udq = 0, \qquad Vdy + W_1dp - Tdq = 0,\\ &Vdx - Rdp + Q_2dq = 0, \qquad Tdx + W_2dy + Udp = 0. \end{aligned}$$

These and (3) yield therefore

$$\begin{aligned} S(VYdx - PUdq) &= U(PRdp + QTdq) + V(TXdx + RYdy)\\ &= PU(Vdx + Q_2dq) + QTUdq + VTXdx - VY(Q_1dx + Udq)\\ &= V(PU + TX - Q_1Y)dx + U(QT + PQ_2 - VY)dq\\ &= -(Q_1 + Q_2)(VYdx - PUdq). \end{aligned}$$

That is to say $Q_1+Q_2=-S$. Also, from (6), it follows that

$$\begin{aligned} Q_1dxdp &= -(Rdy + Udq)dp = -(Rdp)dy - (Udp)dq\\ &= -(Vdx + Q_2dq)dy + (Tdx + W_2dy)dq\\ &= (Tdq - Vdy)dx + (W_2 - Q_2)dydq = W_1dxdp + (W_2 - Q_2)dydq. \end{aligned}$$

Then, on equating the coefficients of $dxdp$ and $dydq$, one sees that

$$Q_1 = W_1, \qquad Q_2 = W_2, \qquad S = -(W_1 + W_2).$$

When Q_1 and Q_2 are replaced by W_1 and W_2, respectively, in (6), it readily follows from the eliminant of the differential terms that $(RT+UV-W_1W_2)^2=0$. Then, inasmuch as

$$W_1 + W_2 = -S, \qquad W_1W_2 = RT + UV,$$

W_1 and W_2 must be the roots of the quadratic, $W^2+SW+(RT+UV)=0$.

It is worthy of note at this point that, when it is remembered that $Q_1=W_1$ and $Q_2=W_2$ in (5), P, Q, X, and Y in (5) are replaced by dp, dq, dx, and dy, respectively, in (6). Hence, if a matrix μ be defined as

$$(7)\qquad \left\| \begin{matrix} -R & W_2 & V & 0\\ W_1 & -T & 0 & V\\ U & 0 & T & W_2\\ 0 & U & W_1 & R \end{matrix} \right\| \equiv \mu,$$

equations (5) and (6) may now be put into the forms,

$$(8)\qquad \mu \cdot \left\| \begin{matrix} dp\\ dq\\ dx\\ dy \end{matrix} \right\| = 0, \qquad \mu \cdot \left\| \begin{matrix} P\\ Q\\ X\\ Y \end{matrix} \right\| = 0.$$

In seeking intermediate integrals or solutions of (1), it is frequently necessary to add to the first four of equations (8) the identity $dz=pdx+qdy$. It is readily

shown also that, from (4) and (5),

$$\frac{R}{Qdx - Xdq} = \frac{T}{Ydp - Pdy} = \frac{U}{Xdy - Ydx} = \frac{V}{Qdp - Pdq} = \frac{W_1}{Ydq - Qdy} = \frac{W_2}{Pdx - Xdp}.$$

The extension of this method consists, of course, in replacing (2) by their more general counterparts, where the number of independent variables is greater than two, and following through the above reasoning.

To consider a particular case, suppose the given equation is

$$R_1u_{zz} + R_2u_{yy} + R_3u_{xx} + S_1u_{yz} + S_2u_{xz} + S_3u_{xy} = V \tag{9}$$

$$u_x \equiv \frac{\partial u}{\partial x}, \qquad u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y}, \text{ etc.,}$$

where R_1, R_2, R_3, S_1, S_2, S_3, and V are functions of x, y, z, u_x, u_y, and u_z. The method proposed therefore depends upon the assumption that (9) is derived from a linear combination of the equations,

$$\begin{aligned} u_x dx + u_y dy + u_z dz &= du, \\ u_{xx}dx + u_{xy}dy + u_{xz}dz &= du_x, \\ u_{xy}dx + u_{yy}dy + u_{yz}dz &= du_y, \\ u_{xz}dx + u_{yz}dy + u_{zz}dz &= du_z. \end{aligned} \tag{10}$$

Suppose therefore that the second, third, and fourth of equations (10) are multiplied by R_1, MR_2, NR_3, respectively, and added. The result is

$$\begin{aligned} &R_1u_{xx}dx + MR_2u_{yy}dy + NR_3u_{zz}dz \\ &\quad + (R_1dy + MR_2dx)u_{xy} \\ &\quad + (MR_2dz + NR_3dy)u_{yz} + (R_1dz + NR_3dx)u_{xz} \\ &\quad = R_1du_x + MR_2du_y + NR_3du_z. \end{aligned}$$

By multiplying (9) by dx and comparing the two results, one concludes that

$$\begin{aligned} dx &= Mdy = Ndz, & S_1dx &= MR_2dz + NR_3dy, \\ S_2dx &= R_1dz + NR_3dx, & S_3dx &= R_1dy + MR_2dx, \\ Vdx &= R_1du_x + MR_2du_y + NR_3du_z. \end{aligned} \tag{11}$$

From these equations, the two pairs,

$$\begin{aligned} (S_2 - NR_3)dx &= R_1dz, & (S_3 - MR_2)dx &= R_1dy, \\ dx &= Ndz, & dx &= Mdy, \end{aligned}$$

then require that

$$R_2M^2 - S_2M + R_1 = 0, \qquad R_3N^2 - S_2N + R_1 = 0.$$

From these equations M and N may be obtained therefore, but these are subject to a restriction imposed by a third equation obtained from the first line of the equations in (11). This is, the values of M and N used must satisfy

$$R_2\left(\frac{M}{N}\right)^2 - S_1\left(\frac{M}{N}\right) + R_3 = 0.$$

In general, two values of M and N can therefore be used in (11) in seeking possible integrals of (9).

Admittedly, equation (9) is a very simple one to be used as an example. Perhaps it suffices however, to justify the claim made in the last sentence of the first paragraph.

A NOTE ON SUMS OF POWERS OF INTEGERS

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The identity

$$\sum_1^n r^3 = \left(\sum_1^n r\right)^2 \tag{1}$$

is well known, and the question naturally arises, for what positive integers k, m, p, q does the identity

$$\left(\sum_1^n r^k\right)^p = \left(\sum_1^n r^m\right)^q \tag{2}$$

hold? Clearly no cases of interest will be lost by supposing that $p>q$ and that p and q are coprime; and with these assumptions it is easy to show that (1) is the only solution of (2).

The proof depends on the well-known facts that $\sum_{r=1}^n r^m$ is a polynomial in n, of degree $m+1$, and that the coefficient of n^{m+1} is $1/(m+1)$. Suppose then that (2) holds, *i.e.*, that

$$\left(\frac{n^{k+1}}{k+1} + \cdots\right)^p = \left(\frac{n^{m+1}}{m+1} + \cdots\right)^q,$$

with p and q as described above. Then $p(k+1)=q(m+1)$, $(k+1)^p=(m+1)^q$. Thus $(m+1)^q=(p/q)^q(k+1)^q$, whence $(k+1)^{p-q}=(p/q)^q$, and the R.H.S. of this is an integer because the L.H.S. is. Thus $q=1$, and $(k+1)^{p-1}=p$. But $k+1\geqq 2$; therefore $2^{p-1}\leqq p$, so that $p<3$. Since $p>q$, we have $p=2$, $k=1$, $m=3$.

Reference

1. Sheila M. Edmonds, Math. Gaz., vol. 41, 1957, pp. 187–188.

THE REMAINDER TERM IN NUMERICAL INTEGRATION FORMULAS

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Numerical integration formulas give a linear combination of function values on a finite set of points to approximate a definite integral. The coefficients are chosen so that the formula is exact whenever the function f is a polynomial of sufficiently small degree. The operation may evidently be written as a Stieltjes integral having integrand f, whose integrator has jump discontinuities corresponding to the coefficients of the formula. The exact value of the integral is similar in form, so the remainder may be written as a Stieltjes integral:

$$R(f) = -\int_a^b f(x)dG_0(x). \tag{1}$$

The limits a and b are chosen so that $[a, b]$ includes the range of the definite integral and the points involved in the numerical formula. G_0 is chosen so that $G_0(a)=0$. If $R(f)=0$, when $f(x)\equiv 1$, then $G_0(b)=0$, and integration by parts* gives:

$$R(f) = \int_a^b f'(x)G_0(x)dx. \tag{2}$$

Suppose m is the largest integer such that $R(f)=0$ for every polynomial of degree $\leqq m$. We may define functions G_i, $i\leqq m$ inductively by:

$$G_i(x) = -\int_a^x G_{i-1}(t)dt. \tag{3}$$

We may then integrate (2) by parts repeatedly (this time within the context of Riemann integration) to obtain,† if $i\leqq m$:

$$R(f) = \int_a^b f^{(i+1)}(x)G_i(x)dx. \tag{4}$$

In proving (4) inductively we use the fact that $R(f)=0$, if $f=1, x, \cdots, x^m$. Then if (4) is true for any integer $j<m$, and we set $f(x)=x^{j+1}/(j+1)!$, (4) gives $0=G_{j+1}(b)$, and so the outside terms in the integration by parts formula are zero.

It sometimes happens that $G_m(x)$ does not change sign in $[a, b]$. In this case, the mean value theorem permits an estimate that may be more convenient:

$$R(f) = f^{m+1}(\xi)\int_a^b G_m(x)dx. \tag{5}$$

It is not obvious that $G_i(x)$ does not change sign even in simple particular

* We assume that f is sufficiently smooth to justify these manipulations.

† For a different derivation of this form of the remainder, see [1], p. 161 ff.

cases. In particular cases it is usually easy to observe the number of sign changes of $G_1(x)$ in $[a, b]$. But by the mean value theorem $G_{i-1}(x)$ has at least one more sign change than $G_i(x)$, since $G_i(a) = G_i(b) = 0$. So $G_1(x)$ has at least $m-1$ more sign changes than $G_m(x)$, and if $G_1(x)$ has only $m-1$ sign changes, then $G_m(x)$ has no sign changes.

Consider for example the approximation of $\int_0^3 f(x)dx$ by the value $\frac{3}{8}[f(0)+3f(1)+3f(2)+f(3)]$. The kernel G_0 chosen to satisfy (1) is:

$$G_0(x) = \begin{cases} 0, & \text{if } x = 0, \text{ or } x = 3; \\ \frac{3}{8} - x, & \text{if } 0 < x < 1; \\ \frac{3}{2} - x, & \text{if } 1 \leqq x \leqq 2; \\ \frac{21}{8} - x, & \text{if } 2 < x < 3. \end{cases}$$

It is evident that G_0 is an odd function of $x - \frac{3}{2}$. The succeeding G_i's are alternately even and odd functions of $x - \frac{3}{2}$, and need be calculated only to the midpoint of the interval. $G_1(x)$, calculated by (3) is

$$G_1(x) = \begin{cases} -\frac{3}{8}x + \frac{1}{2}x^2, & 0 \leqq x \leqq 1; \\ \frac{9}{8} - \frac{3}{2}x + \frac{1}{2}x^2 & \text{if } 1 \leqq x \leqq 2. \end{cases}$$

In particular, $G_1(1) = \frac{1}{8}$ and $G_1(\frac{3}{2}) = 0$. Then G_1 changes sign twice in (0, 3) (see Fig. 1). Since $m = 3$ in this case, G_3 does not change sign, and the estimate (5) is valid.

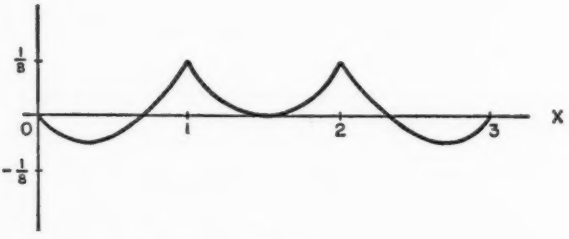

FIG. 1

Hildebrand ([1], p. 73) gives a list of Newton-Cotes formulas for $\int_0^{nh} f(x)dx$, using $f(0)$, $f(h)$, $\cdots$, $f(nh)$, for $1 \leqq n \leqq 8$. In these cases, the number of sign-changes of G_1 is $m-1$, with the exception of the six-point formula,* $n = 5$. In the case of Weddle's rule, six sign changes appear in G_1, and so G_1 could have two sign changes. In this case G_5 really does have two sign changes, and an error estimate of the form (5) is not possible. A somewhat more complicated estimate is given by Hildebrand ([1], p. 160).

* It is known that the remainder estimate (5) is valid for all of the Newton-Cotes formulas. However, the proof is not especially easy (see [2], p. 154 ff., for the case n even and [3] for the case n odd).

References

1. F. B. Hildebrand, Introduction to Numerical Analysis, New York, 1956.
2. J. F. Steffensen, Interpolation, Baltimore, 1927.
3. A. Walter, Zur numerischen Integration, Skand. Aktuarietidskr., 1925.

ELEMENTARY UNIQUENESS THEOREMS FOR DIFFERENTIAL EQUATIONS

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A first course in differential equations probably is not the place to carry through detailed proofs of existence theorems—nor is it necessary to do so since, for the most part, only equations whose solutions can be exhibited are considered. Nevertheless, it is desirable to have uniqueness theorems available to the student in order that he might make use of the many results which depend upon the unique determination of functions by differential equations and initial conditions.

Our purpose is to present such theorems for first and second order linear equations which are elementary in nature and which do not depend upon any particular procedure for the construction of solutions.

The method we discuss uses an inequality which is well known (R. E. Bellman, *Stability Theory of Differential Equations*, New York, 1953) but its derivation is included for the sake of completeness. The method itself is used extensively in the book mentioned for different purposes.

We will assume once and for all the conditions necessary for the existence of the integrals which appear in the calculations.

LEMMA. *If $u(t)$ and $v(t)$ are nonnegative, if k is a positive constant, and if*

$$u(t) \leqq k + \int_0^t u(x)v(x)dx, \tag{1}$$

then,

$$u(t) \leqq k \exp\left(\int_0^t v(x)dx\right). \tag{2}$$

Proof. From (1) it follows that

$$\frac{u(t)v(t)}{k + \int_0^t u(x)v(x)dx} \leqq v(t).$$

By integrating from 0 to t, we have

$$\ln\left(k + \int_0^t u(x)v(x)dx\right) - \ln k \leqq \int_0^t v(x)dx.$$

Then (2) follows immediately.

We first consider the homogeneous, linear, first-order equation,

$$y'(t) + a(t)y(t) = 0, \tag{3}$$

with initial condition $y(0) = y_0$. Integrating (3) from 0 to t, we obtain

$$y(t) = y_0 - \int_0^t a(x)y(x)dx.$$

Therefore,

$$|y(t)| \leqq |y_0| + \int_0^t |a(x)|\,|y(x)|\,dx.$$

This last inequality is of the form (1) so that we may apply (2) and write

$$|y(t)| \leqq |y_0| \exp\left(\int_0^t |a(x)|\,dx\right). \tag{4}$$

This result enables us to prove the following theorem.

THEOREM 1. *If $a(t)$ is continuous, then if for $t \geqq 0$ there exists a continuous solution of the differential equation*

$$u'(t) + a(t)u(t) = f(t) \tag{5}$$

satisfying

$$u(0) = u_0, \tag{6}$$

that solution is unique.

Proof. Suppose that there are two solutions, $u_1(t)$ and $u_2(t)$, satisfying (5) and (6). Then the function $y(t) = u_1(t) - u_2(t)$ satisfies (3) and the initial condition $y(0) = 0$. By (4), $|y(t)| \leqq 0$, which implies $y(t) \equiv 0$. Therefore $u_1(t) \equiv u_2(t)$.

In discussing the second order equation, it will simplify calculations to consider the "normal" form,

$$y'' + a(t)y = 0. \tag{7}$$

Not much generality is sacrificed since the transformation $y = v \exp(-\frac{1}{2}\int a_1(t)dt)$ will reduce $y'' + a_1(t)y' + a_2(t)y = 0$ to an equation of the form (7).

We will assume a continuous solution of (7) satisfying the initial conditions $y(0) = y_0$, $y'(0) = y_1$. Now,

$$y(t) - ty_1 - y_0 = \int_0^t \left(\int_0^x y''(u)du\right)dx = -\int_0^t \left(\int_0^x a(u)y(u)du\right)dx.$$

Integrating the last iterated integral by parts, we obtain,

$$\int_0^t \left(\int_0^x a(u)y(u)du\right)dx = t\int_0^t a(u)y(u)du - \int_0^t xa(x)y(x)dx.$$

Changing the variable of integration from u to x in the first integral on the right hand side of this equation, we may write

$$\int_0^t \left(\int_0^x a(u)y(u)du \right) dx = \int_0^t (t - x)a(x)y(x)dx.$$

Equation (7) is then equivalent to

$$y(t) = y_0 + ty_1 - \int_0^t (t - x)a(x)y(x)dx.$$

Hence, for $t \geqq 0$,

$$|y(t)| \leqq |y_0| + t|y_1| + \int_0^t (t - x)|a(x)||y(x)|dx.$$

If $0 \leqq t \leqq 1$,

$$|y(t)| \leqq |y_0| + |y_1| + \int_0^t (t - x)|a(x)||y(x)|dx,$$

and therefore, from the lemma,

$$|y(t)| \leqq (|y_0| + |y_1|) \exp\left(\int_0^t (t - x)|a(x)|dx \right). \tag{8a}$$

On the other hand, if $t > 1$,

$$|y(t)| \leqq t(|y_0| + |y_1|) + t\int_0^t |a(x)||y(x)|dx,$$

which we may write as

$$\frac{|y(t)|}{t} \leqq |y_0| + |y_1| + \int_0^t x|a(x)| \frac{|y(x)|}{x} dx.$$

An application of the lemma then gives us

$$|y(t)| \leqq t(|y_0| + |y_1|) \exp\left(\int_0^t x|a(x)|dx \right). \tag{8b}$$

On the basis of (8a) and (8b) and in a manner entirely analogous to the first order case, we may now prove the following theorem.

THEOREM 2. *If $a(t)$ is continuous and if for $t \geqq 0$ there is a continuous solution of the equation $u''(t) + a(t)u(t) = f(t)$ satisfying $u(0) = u_0$ and $u'(0) = u_1$, then that solution is unique.*

A MECHANICAL DEVICE FOR FINDING THE REAL ROOTS OF THE CUBIC

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The real roots of the reduced cubic

$$x^3 + ax + b = 0 \tag{1}$$

are given graphically by the intersection of

$$y = x^3, \tag{2}$$

$$y = -ax - b. \tag{3}$$

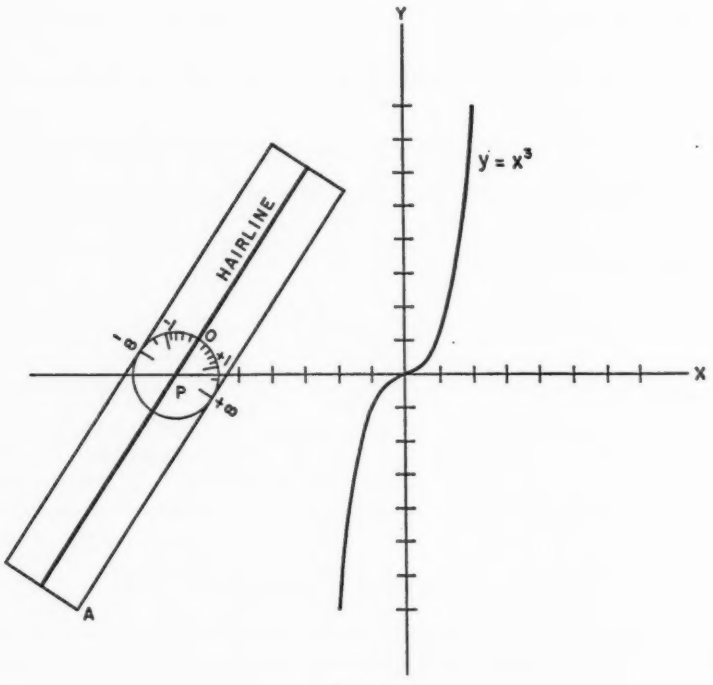

FIG. 1

The mechanical device is designed as follows (Fig. 1): On a plastic strip or one of other suitable material the curve $y=x^3$ is etched. Using a transparent strip, part A is constructed with a pin at its center P which slides along the slotted X-axis; also, A is free to rotate about P. The etchings along the circumference of the circle give the slope of the hairline relative to the X-axis for any position of the hairline other than the vertical. The X and Y axes are calibrated in some convenient units, for example from -10 to $+10$.

Typical operation of the device is as follows: Let $a=-2$, $b=5$. Since $-a$ is the slope of the hairline, rotate A until the circumference marking shows a slope of 2; then slide A parallel to itself until the hairline passes through the point $(0, -5)$; check the slope reading, and make any final adjustments to be sure the readings give slope 2 and Y-intercept -5. Then read off the abscissa of the real intersection of (2) and (3).

To insure that the intersection of (2) and (3) always falls within a readable portion of the graph for large numerical values of slope or Y-intercept, or for small values of both, proceed as follows:

If the roots of (1) are multiplied by m, the resulting equation is

$$x^3 + am^2x + bm^3 = 0,$$

which is equivalent graphically to the intersection of (2) and

$$y = -am^2x - bm^3. \tag{4}$$

For example, if $|-a|$ is too large, multiply it by a suitable fraction, such as $\frac{1}{2}$, and the new line is $y=-\frac{1}{4}ax-\frac{1}{8}b$ with a slope $\frac{1}{4}$ of the original slope and Y-intercept $\frac{1}{8}$ of the previous Y-intercept. This procedure will bring the intersection of the two curves nearer to the two axes. When the point of intersection of (2) and (4) is graphically estimated, multiplication of the resulting roots by $1/m$, 2 in this case, will yield the real roots of (1).

ON PERMUTATIONS AND COMBINATIONS*

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1. In an earlier note Mullin,† gave the following canonic generators for $\Psi(n)$ and $\Phi(n)$:

$$\Psi(n+1) = 2\Psi(n), \qquad \Phi(n+1) = (n+1)\Phi(n) + 1, \tag{1}$$

where

$$\Psi(n) = \sum_{r=0}^{n} {}^nC_r, \quad {}^nC_r = \frac{n!}{(n-r)!r!}, \quad \Phi(n) = \sum_{r=0}^{n} {}^nP_r, \quad {}^nP_r = \frac{n!}{(n-r)!}.$$

He also proved the following results

$$\Phi(n) \sim (n!)\cdot e, \qquad \Phi(n) \sim n^n e^{1-n}\sqrt{(2\pi n)}, \tag{2}$$

where $n \gg 1$ and $e = 2\cdot 718\cdots$. Here we should like to present some canonic generators for finite series involving ${}^nC_r$ and some asymptotic expressions for finite series involving ${}^nP_r$.

2. In most textbooks on college algebra we notice the following properties of the binomial coefficients:

* Through an oversight on the part of the Editor, this note appears without having been proofread by the author.

† A. A. Mullin, Three theorems on permutations, this MONTHLY, vol. 64, 1957, pp. 669–670.

$$\Psi_1(n) \equiv \sum_{r=0}^{n} (2r+1)\,{}^nC_r = (n+1)2^n,$$

$$\Psi_2(n) \equiv \sum_{r=0}^{n} (r+1)\,{}^nC_r = (n+2)2^{n-1},$$

$$\Psi_3(n) \equiv \sum_{r=1}^{n} r\cdot{}^nC_r = n2^{n-1},$$

$$\Psi_4(n) \equiv \sum_{r=0}^{n} {}^nC_r/(r+1) = (2^{n+1}-1)/(n+1),$$

$$\Psi_5(n) \equiv \sum_{r=1}^{n} r\cdot\frac{{}^nC_r}{{}^nC_{r-1}} = \tfrac{1}{2}n(n+1),$$

$$\Psi_6(n) \equiv \sum_{r=0}^{n} ({}^nC_r)^2 = (2n!)/(n!)^2.$$

We therefore easily obtain the following canonic generators for $\Psi_i(n)$, where $i=1, \cdots, 6$.

$$\Psi_1(n+1) = 2\Psi_1(n) + 2^{n+1},$$

$$\Psi_2(n+1) = 2\Psi_2(n) + 2^n,$$

$$\Psi_3(n+1) = 2\Psi_3(n) + 2^n,$$

$$(n+2)\Psi_4(n+1) = 2(n+1)\Psi_4(n) + 1 \quad \text{or} \quad \Psi_4(n+1) \sim 2\Psi_4(n), \quad \text{where } n \gg 1,$$

$$n\Psi_5(n+1) = (n+2)\Psi_5(n) \quad \text{or} \quad \Psi_5(n+1) \sim \Psi_5(n), \quad \text{where } n \gg 1,$$

$$(n+1)\Psi_6(n+1) = 2(2n+1)\Psi_6(n) \quad \text{or} \quad \Psi_6(n+1) \sim 4\Psi_6(n), \quad \text{where } n \gg 1,$$

We also notice that $2\Psi_2(n+1) - 2^2\Psi_2(n) = \Psi_1(n+1) - 2\Psi_1(n)$.

3. Now we consider the expression

$$\Phi'(n) \equiv \sum_{r=0}^{n} (-)^{n-r}\cdot{}^nP_r = n!\left[\frac{(-)^n}{n!} + \frac{(-)^{n-1}}{(n-1)!} + \cdots - \frac{1}{1!} + 1\right].$$

Then

$$\Phi'(n) \sim (n!)e^{-1} \quad \text{when} \quad n \gg 1. \tag{3}$$

Further using Stirling's formula we derive from (3), $\Phi'(n) \sim n^n e^{-(n+1)}\sqrt{(2\pi n)}$, when $n \gg 1$. Again we get from (2) and (3)

$$\Phi(n) \pm \Phi'(n) \sim (n!)\cdot(e \pm e^{-1}), \quad \text{when} \quad n \gg 1. \tag{4}$$

From (4) we easily obtain the following asymptotic expressions

$$\sum_{r=0}^{n/2} {}^nP_{2r} \sim (n!)(e + e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is even};$$

$$\sum_{r=1}^{(n+1)/2} {}^{n}P_{2r-1} \sim (n!)(e + e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is odd};$$

$$\sum_{r=0}^{(n-1)/2} {}^{n}P_{2r} \sim (n!)(e - e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is odd};$$

$$\sum_{r=1}^{n/2} {}^{n}P_{2r-1} \sim (n!)(e - e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is even}.$$

4. We again notice that $r \cdot {}^{n-1}P_{r-1} = {}^{n}P_{r} - {}^{n-1}P_{r}$. Putting $r = 1, \cdots, n$ and adding, we have (assuming that ${}^{n-1}P_{n} \equiv 0$)

$$\begin{aligned} \sum_{r=1}^{n} r \cdot {}^{n-1}P_{r-1} &= \sum_{r=1}^{n} {}^{n}P_{r} - \sum_{r=1}^{n} {}^{n-1}P_{r}, \\ &= \sum_{r=1}^{n} {}^{n}P_{r} - \sum_{r=1}^{n-1} {}^{n-1}P_{r} = \Phi(n) + \Phi(n-1). \end{aligned} \tag{5}$$

Thus when $n \gg 1$ we have from (2) and (5):

$$\theta(n) \equiv \sum_{r=1}^{n} r \cdot {}^{n-1}P_{r-1} \sim \{(n-1)!\} \cdot (n-1)e.$$

Using Stirling's formula, we get $\theta(n) \sim \sqrt{(2\pi)}(n-1)^{n+\frac{1}{2}} \cdot e^{2-n}$. Lastly, to find a canonic generator for $\theta(n)$, we consider (1) and (5) to obtain $\theta(n) = (n-1)\Phi(n-1) + 1$. Now

$$\{\theta(n+1) - 1\}/n = \Phi(n) = n\Phi(n-1) + 1 = \{n/(n-1)\}\{\theta(n) - 1\} + 1.$$

Therefore $(n-1)\theta(n+1) = n^2\theta(n) - 1$, which is a canonic generator for $\theta(n)$.

INVERSION OF JACOBIAN MATRICES*

LOUIS BRAND, University of Houston

1. Jacobian matrices. In a transformation of coordinates

$$x^i = f^i(y^1, y^2, \cdots, y^n), \qquad i = 1, \cdots, n,$$

the n^2 derivatives $\partial x^i/\partial y^j$ may be computed directly and arranged in a square array, the *Jacobian matrix* $(\partial x^i/\partial y^j)$. If the n functions f^i do not satisfy a functional relation their Jacobian matrix is nonsingular and admits a reciprocal. Using the summation convention

$$\frac{\partial x^i}{\partial y^r}\frac{\partial y^r}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j, \qquad i, j = 1, \cdots, n;$$

* Presented to the Texas Section of the Association, Austin, April 17, 1959.

hence if we agree to write Jacobian matrices so that the uppermost index denotes a row,

$$\left(\frac{\partial y^i}{\partial x^j}\right)\left(\frac{\partial x^i}{\partial y^j}\right) = I_n, \tag{1}$$

where I_n is the unit matrix of order n. Therefore the n^2 derivatives $\partial y^i/\partial x^j$ may be computed by inverting the matrix $(\partial x^i/\partial y^j)$.

2. Orthogonal transformations. The inversion of the Jacobian matrix $(\partial x^i/\partial y^j)$ is very simple in the important case of an orthogonal transformation of coordinates. Then we can always write the Jacobian matrix as the product MD of an orthogonal matrix and a diagonal matrix:

$$\left(\frac{\partial x^i}{\partial y^j}\right) = MD. \tag{2}$$

Here D is the diagonal matrix with the positive elements $d_1, \cdots, d_n$, where

$$d_i^2 = \left(\frac{\partial x^1}{\partial y^i}\right)^2 + \cdots + \left(\frac{\partial x^n}{\partial y^i}\right)^2,$$

and M is the orthogonal matrix obtained by dividing the columns of $(\partial x^i/\partial y^j)$ by $d_1, \cdots, d_n$ respectively. The inverted Jacobian matrix is now

$$\left(\frac{\partial y^i}{\partial x^j}\right) = (MD)^{-1} = D^{-1}M^{-1} = D^{-1}M', \tag{3}$$

where M' is the transpose of M and D^{-1} is the diagonal matrix with the elements $1/d_1, \cdots, 1/d_n$.

Consider for example the change from rectangular to spherical coordinates:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Then with $x^i = x, y, z$, $y^i = r, \theta, \phi$, we have $d_1 = 1$, $d_2 = r$, $d_3 = r\sin\theta$, and the matrix

$$\begin{pmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix}.$$

Therefore the matrix

$$\begin{pmatrix} r_x & r_y & r_z \\ \theta_x & \theta_y & \theta_z \\ \phi_x & \phi_y & \phi_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/(r\sin\theta) \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

from which all nine derivatives on the left may be read off.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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SUMMARY OF CONTENT OF SMSG COURSES

Each of the chairmen of writing teams for the SMSG mathematics courses, grades 7 through 12, was asked to prepare a brief summary statement of the content of the new course and ways in which the new course differs from the traditional. These statements are given in the following paragraphs.

Grades 7 and 8. Traditional mathematics courses for grades 7 and 8 include a review of the operations with whole numbers, fractions, and decimals. Percent is introduced, usually in terms of the three cases of percents, each of which is treated separately after various manipulations with percents, including fractional and decimal equivalents of percents. The traditional courses also have rather extensive treatments of percent applications such as commission, simple interest, discount, and insurance. A study of measurement has had an important place, but again much of this is a review of work done in earlier grades and little or none of it is new from a mathematical point of view.

While the new SMSG courses provide for review of the fundamentals of arithmetic, this review has been placed in a new setting with emphasis on number systems. Number systems are treated from an algebraic viewpoint, not only to deepen the student's understanding of arithmetic but also to prepare him for the algebra which is to come. The work on fractions is introduced by defining a fraction as a numeral for the rational number a/b such that $b(a/b)=a$, $b\neq 0$. The grade 8 text starts with an informal treatment of coordinates and equations, and includes a brief introductory chapter on probability. Some of the probability problems were written by biologists associated with the Biological Sciences Curriculum Study, and in a chapter on the lever an attempt has been made to use language consistent with that of the Physical Science Study Committee physics course. Percent applications have a place in the new courses, as do other social applications, for example through governmental statistics in the chapter on graphs and in probability.

The new courses give more than one third of the time to geometry, which is a very considerable change in emphasis from the traditional. Geometric ideas are introduced, first of all, from a nonmetric point of view and then, after a careful treatment of measurement, students are led gradually to a study of properties of triangles and other geometric figures, plane and solid, through an informal deductive approach. Although there is no attempt to give a system of postulates for the geometry, properties are identified on an intuitive or inductive basis and then these properties are used to draw conclusions about, or to prove, other properties. In the chapter on drawings and constructions, instruments in addition to the classical ones are introduced and the student is also provided with experience in sketching figures, especially three-dimensional figures. A grade 8 chapter on nonmetric geometry which comes just before the study of the measurement of volumes and surface areas is, in its topological approach, one of the greatest innovations.

Grade 9. The SMSG ninth-grade course, *First Course in Algebra*, differs from conventional texts in the following ways. It is based upon structure properties of the real number system. This development of algebra is interesting, meaningful, and mathematically sound. It helps bring out the nature of mathematics and strengthens the student's algebraic techniques by relating them to basic ideas. Definitions and properties

are carefully formulated and there is some work with simple proofs. The reading material, which is an important part of the course, is designed to help the student discover ideas. The number line and the simpler language of sets are used to help express the ideas. Inequalities are treated along with equations.

However, as its title suggests, the SMSG ninth-grade course covers essentially the same material as does a conventional first-year algebra text. It teaches the student how to perform the fundamental operations with real numbers and with variables and how to do the usual algebraic manipulations, including factoring, powers and roots, and work with polynomials and fractional expressions. It shows how to solve equations up through quadratic equations in one variable and linear equations in two variables. Graphs of linear and quadratic functions are treated. There is much experience in solving word problems.

Grade 10. The SMSG geometry text differs from conventional ones in content, postulational scheme and manner of treatment.

1. No artificial distinction is made between plane and solid geometry, and a considerable amount of the latter is included. Also, an introduction to analytic plane geometry is provided.
2. The postulate system is a modification of Birkhoff's and is complete. Real numbers are used freely throughout the text, both in the theory and in problems.
3. Accuracy in the statement and use of postulates, definitions and theorems is emphasized.

On the other hand, the text is still basically a treatment of synthetic Euclidean geometry, covering the usual topics: congruence, similarity, parallelism and perpendicularity, area, circles, and construction with straight-edge and compass. There is a main sequence of proved theorems, some minor stated theorems with proofs left as exercises for the students, and a long list of "originals." The basic postulates, definitions and theorems are motivated by appeal to intuition, and many practical and computational problems are given.

Grade 11. The SMSG eleventh-grade text differs from traditional texts in the following important ways:

1. The SMSG text makes much greater demands on the student's ability to learn by reading carefully worded expositions. The writers believe that the development of this ability is essential for success in college mathematics.
2. The study of number systems is stressed as the basis for all understanding of both elementary and advanced mathematics.
3. The idea that algebra is a logical structure built on a relatively small number of fundamental principles is emphasized throughout the text.
4. Presentations which lead the student to make certain predetermined "discoveries" are used where appropriate.
5. Proof is emphasized throughout in order that the student may gain some idea of the nature of a valid mathematical argument.
6. The function concept is developed spirally throughout the text.
7. Coordinate geometry is introduced earlier than usual and is used as a tool in the development of subsequent sections, notably those on trigonometry.
8. The presentation of logarithms reflects contemporary usage which requires more understanding of logarithms and exponential functions and relatively less emphasis on logarithmic computations.
9. The treatment of trigonometry emphasizes identities, equations and graphs of the trigonometric functions more than the computations required in the solutions of triangles.
10. Vectors are developed as a mathematical system and are applied to the solution of a wide variety of problems.

The writers hope that through the studying of this text the student will acquire some ability to handle unforeseen and unforeseeable problems.

The SMSG eleventh-grade text is similar to conventional texts in these respects.

1. The test begins with a review and extension of the basic skills of first year algebra. This review is included in the initial study of number systems.

2. The content is essentially the same as that found in conventional courses in trigonometry and college algebra.

3. Practical applications are given about the same amount of attention as in conventional texts. It was not possible to increase appreciably the number of applications without making unwarranted assumptions about the student's understanding of related fields.

4. The exposition makes use of many illustrative examples and drawings.

5. There is an abundant supply of exercises which have been carefully graded as to difficulty.

Grade 12. The subject matter of *Elementary Functions* is basically conventional. It includes such topics from the theory of equations as the remainder and factor theorems and the usual methods for finding rational roots. The student will find the laws of exponents and logarithms and the rules for changing the base. The chapter on circular functions contains the familiar addition and subtraction formulas and their consequences, identities and equations, and inverse trigonometric functions. An appendix treats the solution of triangles. Emphasis is laid on graphs. However, each of these topics is treated with some novelty and in a new spirit.

Elementary Functions applies the concept of mapping to polynomial, exponential, logarithmic and circular functions. Effective use is made of the ideas of composition and inversion. The treatment of tangents is intuitive, elementary and rigorous. It permits the introduction of Newton's method, and applications to maximum-minimum problems and to graphing. The treatment prepares for calculus without trespassing upon it. The explanation of exponentials and logarithms is novel and unusually clear and thorough. Trigonometry is freshly developed in line with the mapping idea. The style is informal. Explanations are full and concrete, and they convey the spirit of mathematical thinking.

OFFERINGS FOR FRESHMEN

BANCROFT H. BROWN, Dartmouth College

In 1958 the Committee on the Undergraduate Program of the Mathematical Association of America brought out an "Outline of Recommended Courses" (The Duren Report).

The first recommendation is: "A student who enters college meaning to take a (technical) course in mathematics must be ready without further preparation to take a course in calculus and analytic geometry."

With that statement those of us at Dartmouth responsible for the elementary courses agree wholeheartedly. The emphasis is on the word "calculus." The statement "and analytic geometry" is somewhat ambiguous. The elements of this subject are actually in the secondary course. Many colleges give only lip service to this. The analytic geometry of Salmon, of Loney, and of Fine, a discipline, or perhaps a fine art, studied for its own sake, is a luxury, a pretty-pretty which we can't afford.

The Duren Report defines in detail two courses, Calculus I and Calculus II.

Dartmouth had pretty well anticipated these recommendations by several years, and after some five years of experience, we are able to offer definite comments. The content of I and II together is good, but Calculus I could be improved. The Duren Report makes the clear distinction that the analytic geometry in Calculus I shall be of affine character only. This sounds well in theory, but in practice such a course is too dull, too abstract, and too monotonous. We think we should occasionally go Euclidean, and introduce the sine and cosine, complex numbers, and the solution of the second order linear homogeneous differential equation with constant coefficients. Our experience shows that this does not involve a breathless pace.

More generally, this is why we think the Duren Calculus I needs substantial changes. Many of our students who later will major in the social sciences take Calculus I and follow this with a course on "Elementary Mathematics of Sets," or "Finite Mathematics." Suppose we follow the Duren Report literally. They will have experience with exponential functions, but none with oscillatory functions. It may be true that the social sciences have made little use of oscillatory functions in the past, but can we guarantee that this will be true in the future? It seems to us that the first course (and for many this will be the only course in the calculus) should be as broad as possible. Admittedly you have to steer between breadth and superficiality. We think it can be done.

The difficult problems of honors sections and advanced placement cannot be considered in this brief review. At Dartmouth this year we had 35 freshmen, who had had the equivalent of a college semester or more of calculus whom we put in advanced placement; and we put 60 more men in honors groups. There are no easy answers; these are local problems.

One curious new development should be pointed out, and labeled for the menace that it is. It has become increasingly prevalent for secondary schools to give their seniors brief courses from two weeks up to two months "to show what the calculus is all about." This isn't at all for advanced placement. The idea, apparently laudable but absolutely fallacious, is that this will facilitate the change from high school to college. The truth is that nothing hurts a boy so much as to enter college thinking he knows all about the calculus, believing that he is way ahead of his class and can coast the first month or two. The Day of Reckoning is just as sure as death and taxes. Every college welcomes with open arms the student who can be put in advanced placement; but this half-way stuff is a headache for everyone.

At Dartmouth, nearly three-quarters of the freshman class elect the calculus (regular course, honors, or advanced placement). Of the remaining quarter, some lack ability, many have plenty of ability, but lack interest. Any "requirement," any concerted effort to bring this group into the calculus course would not be to their best interest and would inevitably weaken the course. (In fact, if by some auspicious use of the crystal ball we could persuade the lowest 7 per cent of the present calculus group to stay away, every one would be happier.) Should this remaining 25 per cent, or any substantial part of it elect any mathe-

matics? If it is a technical course, my answer is no. Specifically, I have no faintest interest in patching together assorted topics from college algebra, trigonometry, solid geometry, and analytic geometry. But this 25 per cent must never be sold short; it contains at least 50 per cent of the creative genius of the entering class. Can a nontechnical course be devised for these men which is not a travesty on mathematics or science?

I think we have found an affirmative answer in our elective course "Inquiries in Mathematics," which has been taken by some 2700 students in the last 15 years. The course features number, time, and space. "Number" includes such topics as the number system, numbers to other bases, Fermat's Minor Theorem and its converse, Fechner's Law, and the paradoxes of scoring systems. "Time" includes a study of several calendars, including the highly complicated, and apparently arbitrary Jewish Calendar. "Space" is primarily concerned with cartography. Other topics have at various times included the application of Kepler's Laws to sputniks, Euler's theory of unicursal net-works, and a systematic survey of organized professional gambling. The emphasis is on understanding what the problem is, and then inventing and applying a mathematical technique for its solution.

We think the course has been successful. We don't urge others to adopt it—in fact we have never bothered to write a text. It is a local course, and perhaps I may say, an individual course. But we feel that any college of liberal arts does well to consider that its students who are not interested in science may profit from some unconventional course based largely on scientific curiosity.

SOME COMMENTS ON TEACHING OF THE CALCULUS IN SECONDARY SCHOOLS

GERALD R. RISING, Norwalk Public Schools, Norwalk, Connecticut

In an educational note about college freshmen who have studied the calculus in high school (this MONTHLY, vol. 66, 1959, pp. 584–586), Professor J. H. Neelley of Carnegie Institute of Technology has directed a number of criticisms at secondary teaching which merit serious consideration. These criticisms may be applied to two different types of high school courses: (1) the full-year analytic geometry and calculus course designed usually to meet the requirements of College Entrance Examination Board Advanced Standing in Mathematics; and (2) courses which introduce limits and differentiation and integration of polynomial functions with applications in units ranging from a few weeks to a full semester in length. As a high school teacher of both types of course and as a university teacher of elementary and intermediate calculus, I believe that I can speak with some authority about these programs.

It has been indicated that secondary schools are teaching only mechanical application of formulas without understanding. I believe that the students who have successfully passed the advanced placement type of course in mathematics must know much more than this. The CEEB examination is a fine and sophisticated instrument. Students who have scored high must have done as well as

control groups of college students at good universities. There should be no question about others bypassing this course at the college level. Students who have some background in calculus may well be placed in honors sections in college but should not be considered for further advancement.

The advanced-placement classes which I have seen in New York and in Connecticut have been excellent. College texts are used. In every case with which I am familiar the teacher assigned to this section is the strongest teacher in the department. This may be compared with the staffing often utilized in colleges whereby a graduate student overburdened with his own course work is asked to teach this course. If I were a student, I know which of these teachers I would choose; and my own experience with transfers into my evening college classes from day school and with students pleading for vacation help supports this choice.

There are weak teachers at all levels. I suggest, however, that we cannot retreat from curricular advances because of the weaknesses of individual teachers. I have had students return to visit me who tell me of the rote teaching in their college classes and the failure of teachers on even the professorial level to communicate ideas, but this does not destroy my faith in the general caliber of college instruction. I believe rather that both secondary and college teachers could gain from greater exchange of ideas and teaching methods. I believe, too, that both could benefit greatly from closer inspection of some of the really fine teaching going on in elementary schools.

Certainly the answer to these high-quality students is not, as is so often suggested, more of the same work they have done in secondary schools. This is like the method too often used early in the history of dealing with bright youngsters whereby they were assigned ten more problems of each type than their average neighbor. The students who are in such accelerated college-level programs in high school know their basic mathematics. If they did not, they would not be in such a program. There are today many students going on to college with deficiencies in mathematics as well as in other areas. I do not believe that these are the two to ten per cent taking part in accelerated mathematics programs. From my experience with these children I would gladly match them with second-semester college freshmen.

Here, because I believe that it is appropriate, I will describe the mathematics program which I organized at Greece Olympia High School in Rochester, New York, a program similar to that offered in many fine secondary schools over the country. A student can elect: elementary algebra (in grade 9), synthetic plane geometry with some analytics (10), intermediate algebra and trigonometry integrated (11), and half-year courses in advanced algebra and solid geometry concurrently with a full-year experimental course devoted to many of the non-analysis areas so often discussed as "modern mathematics" (12). Thus a non-accelerated student can elect five years of high school mathematics without infringing on the college domain (unless you recall that secondary schools have taken over both college algebra and trigonometry within the last two decades).

To take advanced placement about five per cent of the students take seventh- and eighth-grade mathematics in the seventh grade, push down the courses listed above one year, and take advanced-placement analytic geometry and calculus in their senior year. It is my contention that students completing this program are in an excellent position to enter sophomore college mathematics courses. This type of student is progressing successfully in some of the best universities.

While it is true that some colleges like Dartmouth and Notre Dame which have unusual freshman programs are not in a good position to accept the advanced-placement students, still the vast majority of schools are teaching the traditional course using any one of a dozen texts which are substantially the same. Mathematics teachers on the secondary level should encourage students to find out the type of course offerings in schools at which the students hope to matriculate. In one case I discouraged a student from accelerating in mathematics because of this difficulty. He was able to devote this extra time to another field. Unfortunately such early decisions are very difficult to make.

It is somewhat more difficult to defend the introduction to calculus given in a semester or less in secondary schools. I, too, disagree with what some schools are offering. I do believe, however, that this type of unit can be of real service to the college program. Students can spend a substantial amount of time on two topics which most college teachers will agree deserve more time than they are able to allot to them. These two topics are limits and the definition of the derivative. Rather than gloss over these topics superficially in a lesson or two, as is so often done in college courses, the secondary program offers the possibility of spending two or three weeks on each topic. I do not believe that it is necessary to teach any formulas in calculus in a secondary program, the definition providing the only basis for taking derivatives. By solid teaching of these topics the secondary teacher can help remove what I believe to be two of the toughest hurdles to mathematics students in the college program.

These students should certainly not be allowed to skip a semester of calculus. It is appropriate for them to dig more deeply into elementary calculus in the sort of honors program offered at some colleges, following approaches like that of Artin or Hardy.

Here are some specific recommendations to the college critics of secondary teaching:

(1) Communicate directly with the secondary teacher or principal of students you think are poorly prepared. In many cases you will find that the real fault lies within the admissions office of your own college. All too often even the best colleges are accepting students who do not deserve to go on to higher education.

(2) Compare your criticisms of the secondary programs with the graduate school's criticisms of your own. We all have the tendency to blame the teacher farther down the line, forgetting that at no level and with no student can we expect absolute mastery.

(3) Try to understand the real point of the accelerated programs. They are designed to give the talented student an opportunity to go farther and more deeply into mathematics on the college level.

(4) Realize that you are not alone in having to revise your program because of the changing education at lower levels. All along the line revisions are forcing higher grade levels to adapt to the changing background of the students. Unfortunately this is a cumulative process, higher grades being forced to revise several times. You at the college level will be hardest hit by this process, but you should also be able to reap the most benefit from it.

A COOPERATIVE PROGRAM FOR TEACHER EDUCATION LEADING TO THE B.A. AND B.S. DEGREES

WILLIAM H. EDSON AND J. W. BUCHTA, University of Minnesota

How to combine the specialization required for beginning graduate work in mathematics and science with the breadth of preparation that is desirable for secondary school teaching is a matter of general concern. Degree requirements of liberal arts colleges are frequently set with graduate school preparation in mind, whereas colleges of education have usually looked to the job of the secondary school teacher in specifying their curriculums.

The College of Education and the College of Science, Literature, and the Arts at the University of Minnesota are working together in the second year of an administrative arrangement that enables students to earn concurrently the B.S. degree granted by the College of Education and the B.A. degree granted by the Arts College. Students graduating under the plan have the course work prerequisite to graduate school admission and are eligible for the teaching certificate in Minnesota.

Under the cooperative plan, students meet *all* of the degree requirements of each college. These include the courses in psychology, professional education and practice teaching in the College of Education, the distribution requirements in language, science, social science, English, and humanities, and the major and minor requirements in the Arts College. Since the liberal arts courses specified by the College of Education at Minnesota occupy the greater fraction of a student's time, it is possible for many, by proper planning, to satisfy the requirements of the two colleges by attending one quarter or a summer and a quarter beyond the four years. The time will depend on the major, the planning of a program early in the college career, and the credit load a student is able to carry.

Although several minor procedural modifications concerning advising, registration, and record keeping were necessary, only two changes affecting degree requirements were made. First, the liberal arts college agreed to accept 13 quarter credits of educational psychology and educational philosophy toward its general elective requirements. Second, the colleges agreed that the residence

requirement in either school could be satisfied by three quarters of registration after admission to the upper division (junior year).

Under the plan students are enrolled in the College of Science, Literature, and the Arts for the first two years, as are all students preparing to teach secondary academic subjects. At the beginning of the junior year those desiring the combined program apply for admission to the Upper Division of the College of Science, Literature, and the Arts and for admission to the College of Education. If approved by both colleges they register in the Upper Division of the liberal arts college during their junior year and in the College of Education in their senior year.

The combined plan has not been designed to replace the regular programs in either of the colleges but rather to attract some additional people to teaching in all fields, especially in science and mathematics, in our secondary schools and to encourage graduate work in the teaching subject area. In this, the plan appears to be successful.

Mathematics for Parents

The parents of junior high school students in the Westport, Connecticut, public schools have expressed considerable interest in the public school program of mathematics since 1958. Their interest has increased steadily, and last year parents began to ask for some way to find out more about the new curriculum. The main reason for this interest seems to be a desire on their part to help their children with the practice work at home.

As a result of this interest, a course for parents is being conducted on Thursday evenings from 7:30 P.M. to 9:30 P.M. through the school's adult education program. The teacher of this class is the Westport mathematics supervisor. Each adult in the course has a seventh grade text (SMSG), 1960 edition, and the plan is to go as far and as fast as the class will permit. The class consists of 24 adults, quite heterogeneous in background. The interest is high and more people have expressed a desire to join the group.

RAY WALCH, Mathematics Supervisor, Westport, Connecticut, Public Schools

A Survey of New Programs in Mathematics

Studies in Mathematics Education, A Brief Survey of Improvement Programs for School Mathematics has recently been prepared by the mathematics department of Scott, Foresman and Company and is available for teachers. The pamphlet contains a concise summary of the work of the various study groups in mathematics and may be especially useful for inservice workshops and teacher training classes. This pamphlet is an extension and revision of an earlier pamphlet prepared in 1959. Descriptions of several programs have been added in this revision. The price of the pamphlet is 50 cents.

California Mathematics and Science Teachers to Meet

The California Mathematics Council and the California Science Teachers Association are planning a joint spring meeting for April 28–30, 1961. The California Mathematics Council is affiliated with the National Council of Teachers of Mathematics and the California Science Teachers Association with the National Science Teachers Association. Meetings of mathematics and science teachers at the secondary school level are not too frequent and California is to be congratulated on promoting this kind of cooperative action. Over 2000 teachers took part in the annual meeting of the California Mathematics Council held in Monterey in December.

Modernized Methods of School Mathematics Teaching

A new approach to the teaching of mathematics in school, to replace the outworn methods of the past, was planned by a working party organized and financed by the O.E.E.C. Office for Scientific and Technical Personnel, which recently met in Zagreb and Dubrovnik in Yugoslavia. This new approach included improved methods of presentation, designed to appeal to the interest of the pupil and at the same time to stress the connections between the various branches of mathematics.

The working party, which consisted of twenty leading mathematicians from O.E.E.C. countries and the United States, included representatives of universities, secondary schools and institutions training secondary school teachers. European countries represented in the group were Belgium, Denmark, France, Germany, Italy, Sweden, Switzerland, the United Kingdom and Yugoslavia.

The modernized curriculum recommended includes new and improved contents for courses in algebra, geometry and analysis, and better integration of subject matter in algebra itself and between algebra and other branches of mathematics. It is designed to aid national authorities in preparing new textbooks and planning regular and experimental courses in mathematics.

This endeavor to substitute a new and livelier spirit in the teaching of mathematics for uninspiring traditional methods forms part of the O.S.T.P. program for stimulating co-operative action among member countries in educating sufficient scientists, engineers and technicians to meet their increasing needs for these specialized skills.

(News Release by OEEC Information Division, October 3, 1960).

Other NSF Visiting Lecturer Programs

Secondary Schools. In addition to the support given by National Science Foundation to MAA for a Visiting Lecturer Program to Secondary Schools, NSF supports visiting lecturer programs to secondary schools sponsored on a national basis by the American Institute of Biological Sciences, American Chemical Society, and the American Institute of Physics.

Colleges. The National Science Foundation supports a visiting lecturer program to colleges which is sponsored by the Society for Industrial and Applied Mathematics, as well as the MAA program reported last month. Other NSF college programs of visiting lecturers are sponsored by the American Anthropological Association; American Society for Engineering Education; Society of American Foresters; American Geological Institute; American Institute of Physics; American Physiological Society; American Psychological Association; Society of Wood Science and Technology, and American Meteorological Society.

Foreign Scientists. The National Science Foundation also sponsors lecturer programs in which the lecturers are scientists from foreign countries. There are nine such programs, including one under the auspices of the American Mathematical Society. John W. Green, secretary of the Society, is director of the program for mathematics. These programs for visiting lecturers are also sponsored by the American Astronomical Society; American Institute of Biological Sciences; American Chemical Society; Engineers Joint Council; American Geological Institute; American Meteorological Society; American Institute of Physics, and American Psychological Association.

Academies of Science. Through a different program of the National Science Foundation, support is given to educational programs in the sciences sponsored by state academies of science, sometimes in cooperation with other local scientific groups, including sections of the Mathematical Association of America. While many of the academies of

science do not have a section on mathematics, others do. Quite a number of their programs which provide for visiting lecturers from colleges and universities to the secondary schools in the states do include mathematics, but the mathematics lecturers are often not very numerous. Among the academy of science programs for 1960–61 which include visiting lecturers as a part of the program, are those sponsored by the state academies in Arizona, Arkansas, Colorado-Wyoming, Indiana, Iowa, Maryland, Minnesota, Montana, Nebraska, New Mexico, Ohio, Texas, Utah, Virginia, and West Virginia. In addition, under the NSF state academies of science project, a grant has been made to the Mathematics Speakers' Bureau of metropolitan New York for visiting lecturers to high schools in their area. The grant is made directly to Cooper Union, New York, since the sponsoring group is not incorporated. The program of the Speakers' Bureau is directed by Professor James N. Eastham of Cooper Union. The regional representatives for the MAA Visiting Lecturer Program to Secondary Schools have been asked to offer full cooperation to the academies of science in their regions so that the programs under MAA auspices and those under the academies can both be strengthened and so that there will not be undesirable duplication.

Supplement to CCSSO Purchase Guide

An updating of the Purchase Guide of 1959 in the form of a 64-page supplement was published in February 1961 by the Council of Chief State School Officers. The supplement was organized on the same plan as their 1959 comprehensive guide, of which some 43,000 copies were distributed without cost to state and local school systems throughout the country. The guide is intended to provide advice on the purchase of new materials for mathematics, science and the modern foreign languages, and to assist teachers and administrators in modernizing course content in curriculum of science and mathematics and in foreign languages.

Science and Mathematics Students Honored at IBM Junior Science Symposium

Three hundred outstanding science and mathematics secondary school students in the metropolitan New York area attended the IBM Junior Science Symposium in October. Co-sponsors with International Business Machines Corporation were the Science Manpower Project of Columbia Teachers College and Columbia University. Dr. Edward Teller, currently Professor of Physics-at-Large at the University of California speaking on "Geometry of Space and Time" at the opening lecture of the symposium, stated, "At the beginning of the century ideas about the world in which we live underwent a radical change. The strangeness of these ideas has prevented their spread. These new concepts should be communicated to our children in high school when their minds are most receptive."

Six of the students, with records of particularly outstanding achievements, presented papers during the symposium on original research they have conducted:

Matrices and determinants, James Pepe, Xaverian High School, Brooklyn, New York.

The use of the digital computer in the investigation of Fibonacci numbers, Harry Saal, Midwood High School, Brooklyn, and Columbia University.

The Stern-Gerlach experiment, Martin Breidenbach, Pascack Valley Regional High School, Hillsdale, New Jersey.

Crystals and their growth, Frank J. Traina, St. Augustine Diocesan High School, Brooklyn, New York.

Isolation and identification of antibiotic-producing microorganisms, Kathleen McGarrity, Aquinas High School, Bronx, New York.

A technique for the quatitative determination of free serum amino acids, Bernard Rappoport, New Dorp High School, Staten Island, New York.

In addition to Dr. Teller, other noted scientists who spoke during the course of the four-day symposium included: Lipman Bers, Institute of Mathematical Sciences, New York University; C. S. Wu, Physics Department, Columbia University; and, M. J. Kopac, Professor of Biology, New York University.

Dean Mina S. Rees of Hunter College conducted a panel discussion on "Tomorrow's Opportunities in Science."

Students were selected by their schools on the basis of their interests and accomplishments in science and mathematics. The symposium is by invitation only and is a distinct honor for those attending. Honored students represented public, private, and parochial secondary schools. The purpose of the symposium is to advance and promote science and mathematics at the secondary school level.

The four-day symposium included a trip to Poughkeepsie, where principal IBM research and product development laboratories and a manufacturing center for medium- and large-scale computers are located. It is the home of STRETCH, a system capable of 75 billion computations a day.

(From an IBM News Release).

CORRECTION

We regret that in the report of the Kentucky Conference of College Science and Mathematics Staff Members (this MONTHLY, vol. 68, 1961, p. 61), the name of Jerrold W. Zacharias was misspelled.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1456. *Proposed by J. F. Darling, Woodstown, N. J.*

Prove that in a triangle with sides a, b, c and semiperimeter s,

$$a^2 + b^2 + c^2 \geqq (36/35)(s^2 + abc/s),$$

with equality only if the triangle is equilateral.

E 1457. *Proposed by Aaron Herschfeld, Pennsylvania State University, Hazleton, Pennsylvania*

Show that the set of numbers $J_m = m^2+1$, $m = 1, 2, \cdots$, contains an infinity of composite $J_N = J_m J_n$. In fact, for arbitrary m, find two pairs of corresponding integers n, N.

E 1458. *Proposed by Hans Schwerdtfeger, McGill University*

Let $x_1 < \cdots < x_n$ be n points on the x-axis. Let P be a point in the (x, y)-plane with ordinate different from zero. If d_i is the distance of P from x_i show that

$$\sum_{j=1}^{n} a_j d_j^2 = \begin{cases} 1 \text{ if } n = 3 \\ 0 \text{ if } n > 3 \end{cases},$$

where $a_i = 1/f'(x_i)$, $f(x) = (x-x_1) \cdots (x-x_n)$.

E 1459. *Proposed by Azriel Rosenfeld, Yeshiva University*

Let $S(m, N)$ be the statement: "There exist m primes not exceeding N which are consecutive terms of an arithmetic progression." For example, $S(3, N)$ is true for $N \geqq 7$(use 3, 5, 7); $S(5, N)$ is true for $N \geqq 29$ (use 5, 11, 17, 23, 29). Prove that $S(7, N)$ is false for $N < 900$, and that $S(11, N)$ is false for $N < 10{,}000$.

E 1460. *Proposed by Masao Arai, Jiyu Gakuen, Tokyo, Japan*

Let x, y, n be positive integers and let $\chi(n)$ denote the number of pairs of integers x, y satisfying $(x, n) = (y, n) = (x+y, n) = 1$, $x < n$, $y < n$. Find a formula for $\chi(n)$.

SOLUTIONS

The Soap Contest

E 1426 [1960, 692]. *Proposed by C. F. Pinzka, University of Cincinnati*

Professor E. P. B. Umbugio is trying to supplement his meager academic salary by entering soap contests. One such contest requires the contestants to find the number of paths in the following array which spell out the word MATHEMATICIAN:

```
                                M
                              M A M
                            M A T A M
                          M A T H T A M
                        M A T H E H T A M
                      M A T H E M E H T A M
                    M A T H E M A M E H T A M
                  M A T H E M A T A M E H T A M
                M A T H E M A T I T A M E H T A M
              M A T H E M A T I C I T A M E H T A M
            M A T H E M A T I C I C I T A M E H T A M
          M A T H E M A T I C I A I C I T A M E H T A M
        M A T H E M A T I C I A N A I C I T A M E H T A M
```

Umbugio has counted 1587 paths which originate from one of the first five rows. With the deadline for submitting entries approaching, he is distraught to say the least. Help the Professor out by finding the number of paths with a minimum of computation.

I. *Solution by J. F. Leetch, Ohio State University.* One may count paths "backwards" from the N. In counting the left half of the array, including the center column, there are two choices for each backward step. Thus, this portion yields 2^{12} paths. Doubling this number and subtracting the center column, which was counted twice, yields $2^{13}-1=8191$ paths.

II. *Solution by D. A. Moran, University of Illinois.* Every path must originate on a "boundary M" and terminate at the unique N. The paths lying completely in the left half of the array (including the vertical path) correspond one-to-one with the words of twelve letters each of which is chosen from the pair (H, V), where H stands for "horizontal" and V for "vertical." The number of these is $2^{12}=4096$. The paths lying completely in the right half of the array (excluding the vertical path) are similarly found to number $2^{12}-1=4095$. Thus the total number of distinct paths is 8191, since every path lies completely in one half of the array.

Also solved by B. C. Anderson, R. H. Anglin, Leon Bankoff, S. D. Barcun, D. Y. Barrer, Alan Beal, J. D. Beyer, A. M. Broshi, M. L. Cantor, Robert Carlos, J. W. Clawson, R. J. Cormier, Dennis Couzin, Rufus Crane, H. H. Crapo, G. S. Cunningham, H. T. David, R. J. Distler, K. P. Dressler, Underwood Dudley, O. E. Eason, S. J. Einhorn, S. H. Eisman, J. S. Elston, Jeanne G. English, E. T. Frankel, J. E. Freund, W. W. Funkenbusch, José Gallego-Diaz, Michael Goldberg, L. D. Goldstone, H. W. Gould, R. E. Greenwood, Stan Guberud, E. W. Harrington, Jr., Charles Hatfield, Frank Hawthorne, R. L. Helmbold, L. J. Huber, Erwin Just, William Kantor, L. F. Klosinski, David Lash, You-Freng Lin, M. V. Mahoney and P. L. Renz (jointly), E. W. Marchand, D. C. B. Marsh, Glenn Mayfield, Ann Miller, Walter Penney, Frank Perez, J. L. Pietenpol, I. N. Presson, E. V. Price, J. L. Purdy, Alvah Raymond, Robin Robinson, L. R. Ruch, T. A. Schoen, Arnold Singer, G. E. Smith, W. B. Stovall, Jr., Eric Sturley, J. D. Vineyard, J. A. Ward, K. P. Yanosko, and the proposer. Late solutions by H. C. Dixon, Jr., Anthony Hug, J. B. Muskat, Adrian Peterson, and Mary Agnes Racki.

Bankoff pointed out that the Professor evidently failed to notice that the number of letters in MATHEMATICIAN and the sum of the digits in E 1426 are both equal to 13! Several solvers called attention to the similar Problems 256, 257, 258 in Dudeney's *Amusements in Mathematics* and Problems 30 and 38 in his *Canterbury Puzzles.*

A Further Triangle Inequality

E 1427 [1960, 692]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If a_1, a_2, a_3 are the sides and h_1, h_2, h_3 the altitudes of a triangle T, show that

$$\sqrt{3}\sum_1^3 a_i \geqq 2\sum_1^3 h_i,$$

with equality if and only if T is equilateral. (Dedicated to the memory of Victor Thébault.)

Solution by Leonard Carlitz, Duke University. We have first

$$(\sum a_i)^2 \geqq 3\sum a_1a_2 = (3a_1a_2a_3/2\Delta)\sum h_i = 6R\sum h_i,$$

where Δ is the area and R the circumradius of the triangle, with equality only in the case of an equilateral triangle. Now, as a special case of a theorem on

convex polygons (see Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und in Raum*, p. 6), we have also $3R\sqrt{3} \geqq \sum a_i$, with equality only in the case of an equilateral triangle. Consequently

$$\sqrt{3}(\sum a_i)^2 \geqq 2(\sum a_i)(\sum h_i),$$

so that $\sqrt{3}\sum a_i \geqq 2\sum h_i$, with equality only in the case of an equilateral triangle.

Also solved by A. N. Aheart, Leon Bankoff, T. R. Curry, Jane Evans, José Gallego-Diaz, L. D. Goldstone, Earl High, William Cantor, D. C. B. Marsh, Norman Schaumberger, P. D. Thomas, Dale Woods, and the proposer. Late solutions by H. E. Bray, C. P. Donahoo, Jr., Bob Snell, and Guy Torchinelli.

An Infinite Product

E 1428 [1960, 693]. *Proposed by Aaron Lieberman, The G. C. Dewey Corporation, N. Y.*

Evaluate

$$\prod_0^\infty [1 + (1/2)^{2^n}].$$

I. *Solution by Leon Bankoff, Los Angeles, Calif.* It is easily verified by induction that

$$\prod_0^n [1 + (1/2)^{2^n}] = 2[1 - (1/2)^{2^{n+1}}].$$

Consequently

$$\lim_{n\to\infty} \prod_0^n [1 + (1/2)^{2^n}] = 2.$$

II. *Solution by D. W. Bailey, University of Oregon.* For $|z| < 1$,

$$\prod_0^\infty [1 + z^{2^n}] = \sum_0^\infty z^n = 1/(1 - z).$$

Also solved by A. N. Aheart, John Avila, Alan Beal, P. B. Bennett and D. P. Kelly and A. L. Sulton (jointly), R. C. Bollinger, A. M. Broshi, W. J. Carpenter, P. R. Chernoff, F. H. Cleveland, Dennis Couzin, Gus Di Antonio, Underwood Dudley, Ragnar Dybvik, S. J. Einhorn, S. H. Eisman, P. G. Engstrom, Jane Evans, F. A. Faucher, J. E. Freund, Stuart Friedman, José Gallego-Diaz, Seymour Geisser and Clifford Patlak (jointly), Michael Goldberg, L. D. Goldstone, Juris Hartmanis, G. A. Heuer, J. E. Homer, Jr. and David Zeitlin (jointly), Erwin Just, William Kantor, J. H. Kaplan, P. G. Kirmser, L. F. Klosinski, A. G. Konheim, Sidney Kravitz, D. C. B. Marsh, Marvin Mielke, R. P. Miller, Otto Mond, C. S. Ogilvy, Walter Penney, J. L. Pietenpol, R. L. Renz, V. M. Sakhave, Norman Schaumberger, George Senge, Arnold Singer, Sister Kenneth Kolmer, Eric Sturley, W. C. Waterhouse, Lawrence Zalcman, and the proposer. Late solutions by D. A. Breault, J. B. Muskat, Dmitri Thoro, and Guy Torchinelli.

Several solvers pointed out that the problem is not new. It is found in Knopp, *Theory of Functions II*, Ex. 4b, p. 21; Ahlfors, *Complex Analysis*, Ex. 2, p. 155; Bromwich, *Infinite Series* (2nd

ed.), Ex. 6, p. 114; Knopp, *Theory and Application of Infinite Series*, Ex. 85b, p. 228; Apostol, *Mathematical Analysis*, Ex. 12–39(d), p. 388.

A Definite Integral

E 1429 [1960, 693]. *Proposed by H. S. Brock, David Taylor Model Basin, Washington, D. C.*

If

$$f(\theta) = \frac{1 + a^2b - 2ab\cos\theta + 2b\cos 2\theta - a(1+b)\cos 3\theta}{1 + a^2 - 2a\cos 3\theta},$$

where $-1<a<1$, show that $\int_0^\pi f(\theta)d\theta = \pi$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$V(n, k) = \int_0^\pi \frac{\cos n\theta d\theta}{1 + a^2 - 2a\cos k\theta} = \begin{cases} \pi a^m/(1-a^2), & \text{if } n = mk, \\ 0, & \text{if } k \nmid n, \end{cases}$$

(see Bromwich, *Introduction to the Theory of Infinite Series*, 2nd ed. (1926), Prob. 26, p. 529), it follows that

$$\int_0^\pi f(\theta)d\theta = (1 + a^2b)V(0,3) - 2abV(1,3) + 2bV(2,3) - a(1+b)V(3,3)$$

$$= [(1 + a^2b)\pi/(1-a^2)] - [a(1+b)\pi a/(1-a^2)] = \pi.$$

Also solved by José Gallego-Diaz, L. D. Goldstone, L. A. MacColl, and D. C. B. Marsh.

Most of the solutions employed contour integration about the unit circle. It is to be noted that the value of the integral is independent of both parameters in the integrand. The proposer said the integral arose in connection with a recently developed theorem in the theory of elasticity.

Double Roots

E 1430 [1960, 693]. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development*

What is the highest order of multiplicity a root can have for the equation

$$x(x-1)(x-2)\cdots(x-n+1) = \lambda?$$

Solution by D. C. B. Marsh, Colorado School of Mines. Since the polynomial $p(x) \equiv x(x-1)(x-2)\cdots(x-n+1)$ has all of its zeros real and distinct, so does $p'(x)$. For $p(x)-\lambda$ to have a zero of multiplicity m, $p'(x)$ must have this zero with multiplicity $m-1$. Therefore any root of $p(x)=\lambda$ can have multiplicity no greater than 2. That 2 is possible is shown by the case $\lambda = p(z)$, where z is a zero of $p'(x)$.

Also solved by D. W. Bailey, Alan Beal, Leonard Carlitz, P. R. Chernoff, Underwood Dudley, José Gallego-Diaz, Seymour Geisser and Clifford Patlak (jointly), Michael Goldberg, R. T. Hood, Erwin Just, William Kantor, A. G. Konheim, R. W. Means, J. L. Pietenpol, P. L. Renz, and W. C. Waterhouse. Late solution by Guy Torchinelli.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problem and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4933 [1960, 927]. CORRECTION. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set-theoretic union of a finite number of commutative fields having the same unit element; prove that R must then be a commutative field.

4953. *Proposed by O. P. Aggarwal and Irwin Guttman, McGill University*

Show that

$$\iint_S e^{-(x^2+y^2)/2}dxdy = \iint_T e^{-(x^2+y^2)/2}dxdy,$$

where S is the square $0 \leqq (x, y) \leqq a$ and T is the triangle bounded by the coordinate axes and the line $x+y=a\sqrt{2}$.

4954. *Proposed by D. J. Newman, Yeshiva University*

Show that, if $f(x) \geqq 0$ is convex, then

$$\int_0^\infty f^2 \leqq \tfrac{2}{3} \max f \int_0^\infty f,$$

and that $\frac{2}{3}$ is best possible.

4955. *Proposed by T. L. Saaty, Office of Naval Research, Washington, D. C.*

Show that

$$\sum_{k=2}^{n} \binom{n}{k} (1-\beta)^k \beta^{n-k} \sum_{j=1}^{k-1} \frac{1}{j} = \sum_{j=1}^{n-1} \frac{(1-\beta^j)(1-\beta^{n-j})}{j}.$$

4956. *Proposed by Leonard Carlitz, Duke University*

Find all analytic solutions $f(x)$, $g(x)$, of the functional equation

$$f(x+y) = f(x)g(y) + f(y)g(x).$$

4957. *Proposed by H. S. Shapiro, New York University*

Let D be a bounded domain in the complex plane, and let $f(z)$, $\phi_1(z)$, $\cdots$, $\phi_n(z)$ be linearly independent functions analytic in and on the boundary of D. Then there exists a unique set of complex numbers $\lambda_1, \cdots, \lambda_n$ such that the following integral is a minimum:

$$\iint_D \left| f(z) - \sum_{k=1}^{n} \lambda_k \phi_k(z) \right| dS.$$

4958. *Proposed by A. G. Konheim, IBM Research, Yorktown Heights, N. Y.*

A polynomial $P(x) = \sum_{i=0}^{n} a_{n-i}x^{n-i}$ will be called symmetric if $a_{n-i} = a_i$, $i = 0, 1, \cdots, n$. Prove that a symmetric polynomial over $GF(2)$ is not primitive.

SOLUTIONS

Maximal Chains in a Boolean Algebra

4871 [1959, 816]. *Proposed by Juris Hartmanis, Ohio State University*

Prove that the Boolean algebra of all subsets of the integers has maximal chains whose length is denumerable and has also maximal chains whose length is not denumerable.

Solution by W. M. Perel and R. C. Wherritt, Louisiana State University, New Orleans. Let I be the set of integers. Define $C_0 = \varnothing$, and, for $n \geqq 1$, let $C_{2n-1} = \{p: |p| \leqq n-1\}$ and $C_{2n} = C_{2n-1} \cup \{n\}$. Then $\{C_m: m \geqq 0\} \cup \{I\}$ is a denumer-maximal chain.

Now let f be any one-to-one function taking the rationals onto the integers. For every real number r, define $D_r = \{f(t): t \text{ rational}, t \leqq r\}$. Then $D_r \subset D_s$ if and only if $r \leqq s$, $D_r \neq D_s$ if $r \neq s$, and $\{D_r: r \text{ real}\} \cup \{\varnothing\} \cup \{I\}$ is a nondenumerable maximal chain. The maximality is equivalent to completeness of the reals in the Dedekind-cut sense.

Also solved by Irma Esrig, J. B. Johnston, Elliott Mendelsohn, and Hans Schneider.

Compact Topological Space

4891 [1960, 188]. *Proposed by I. S. Gál, Yale University*

Let X be a regular topological space and let A be a dense subset of X such that every net with values in A has a nonvoid adherence in X. Show that X is compact. Is regularity necessary?

Solution by the proposer. Let $\{O_i\}$ $(i \in I)$ be an open cover of X which contains no finite subcover. Since X is regular $\{O_i\}$ $(i \in I)$ admits a refinement $\{Q_j\}$ $(j \in J)$ such that * $c(Q_j) \subseteq O_i$ for every $j \in J$ and for some $i = i(j) \in I$. Let the index set I be well ordered and denote by S_k the set $S_k = Q_k - \cup [Q_j: j < k]$. We prove that the interior of S_k is not void for an infinity of indices $k \in J$. For suppose that $S_k^i = \varnothing$ for every $k \geqq \kappa$, that is $c(Q_k) \subseteq c(\cup [Q_j: j < k])$ for every

* For convenience in printing, $c(\ )$ is used to denote closure.

$k \geqq \kappa$, where κ is a finite index. Let $\mathfrak{C}$ be the class of those initial segments $K \subseteq J$ for which

$$c(\cup[Q_j: j \in K]) \subseteq c(\cup[Q_j: j \leqq \kappa]).$$

Then $\mathfrak{C}$ is nonvoid because $(1, \cdots, \kappa) \in \mathfrak{C}$. It can be ordered by inclusion: $K_1 \leqq K_2$ if $K_1 \subseteq K_2$. There is a maximal element in $\mathfrak{C}$ namely $\cup[K: K \in \mathfrak{C}]$ itself is an element of $\mathfrak{C}$. In fact,

$$\cup[Q_j: j \in \cup[K: K \in \mathfrak{C}]] \subseteq c(\cup[Q_j: j \leqq \kappa]),$$

and so the same inclusion relation holds for the closure of this union.

We prove that $\cup[K: K \in \mathfrak{C}]$ is J. For let $K \in \mathfrak{C}$ be a proper subset of J and let k' be the first index not in K. We set $K' = K \cup \{k'\}$ and obtain

$$\begin{aligned} c(\cup[Q_j: j \in K']) &= c(\cup[Q_j: j < k']) \cup c(Q_{k'}) = c(\cup[Q_j: j < k']) \\ &= c(\cup[Q_j: j \in K]) \subseteq c(\cup[Q_j: j \leqq \kappa]). \end{aligned}$$

Hence $K' \in \mathfrak{C}$ and K is not maximal. Consequently $\cup[K: K \in \mathfrak{C}] = J$ and this implies that $\cup[Q_j: j \in J] \subseteq c(\cup[Q_j: j \leqq \kappa])$. However, on the one hand, $\{Q_j\}$ $(j \in J)$ is a cover of X and so $X = c(Q_1) \cup \cdots \cup c(Q_\kappa)$. On the other hand, $\{c(Q_j)\}$ $(j \in J)$ is a refinement of $\{O_i\}$ $(i \in I)$ and by hypothesis $\{O_i\}$ $(i \in I)$ does not contain a finite subcover. This is a contradiction and the original hypothesis is false. In other words, S_k' is not void for infinitely many indices $d \in J$. Let D be any infinite set of such indices $d \in J$, having the property that there is no last element in D. Since A is dense in X the sets $S_d \cap A$ are not void. If $a_d \in S_d \cap A$ for every $d \in D$ then the linear net (a_d) $(d \in D)$ has a void adherence in X. Moreover the sets $\cup[S_d \cap A: d \leqq \delta]$ $(\delta \in D)$ form a filter base in A whose adherence in X is void.

The regularity of X is necessary in the following weak sense: There exists a noncompact X and a dense subset A of X such that every net with values in A has a nonvoid adherence in X. As a matter of fact we can choose X to be a Hausdorff space: Let $X = [-1, +1]$ and let $0 \subseteq X$ be open if it can be obtained from a set which is open in the usual sense in X by omitting points of the form $1/n$ $(n = \pm 1, \pm 2, \cdots)$.

Also solved by Wu Ta-Sun, and J. V. Whittaker.

A Convergent Sequence

4899 [1960, 381]. *Proposed by R. C. Buck, University of Wisconsin*

As $(x, y) \to (0, 0)$ let $\lim F(x, y) = 0$. Suppose that $\{s_n\}$ is a real sequence such that $0 \leqq s_{n+m} \leqq (s_n, s_m)$ for all $n, m = 1, 2, \cdots$. Suppose also that $\lim (s_1 + \cdots + s_n)/n = 0$. Show that $\lim s_n = 0$.

Solution by S. A. Andrea, Oberlin College. Let $\epsilon > 0$ be chosen. Choose δ so that $|x| < \delta$, $|y| < \delta$ implies $|F(x, y)| < \epsilon$. Choose n so that $k \geqq n$ implies $(s_1 + \cdots + s_k)/k < \delta/2$. Then we know that $\sum_{p=1}^{k} (s_p + s_{k+1-p})/k < \delta$. This means

that for some p we have $(s_p+s_{k+1-p})<\delta$ which in turn gives us $s_p<\delta$ and $s_{k+1-p}<\delta$. Now we see that $s_{k+1}\leqq F(s_p, s_{k+1-p})<\epsilon$ for all $k\geqq n$, *i.e.*, $\lim s_n=0$.

Also solved by Robert Breusch, T. C. Brown, E. J. Burr, N. J. Fine, A. J. Geurts, George Glauberman, R. B. Kelman, J. G. Mauldon, M. D. Mavinkurve, J. E. Potter, Albert Wilansky, and the proposer.

Editorial Note. The above proof (as, indeed, in all solutions received) assumes either (a) $F(x, y)$ is continuous at (0, 0) or (b) the s_i are positive. Breusch cites the following counter-example to show that the conclusion does not follow unless some additional assumption is made:

$$F(x, y) = \begin{cases} 1 & \text{if } x = y = 0, \\ x + y & \text{otherwise;} \end{cases} \qquad s_k = \begin{cases} 1 & \text{if } k = 2^r, \\ 0 & \text{otherwise.} \end{cases}$$

Sequence with Arbitrarily Slow Convergence

4900 [1960, 382]. Proposed by S. P. Lloyd, Bell Telephone Laboratories

For every measurable subset E of the unit interval let $\{C_n(E)\}$ denote the sequence of numbers

$$C_n(E) = \int_E e^{2\pi inx}dx, \qquad n = 1, 2, \cdots.$$

Show that a sequence $\{C_n(E)\}$ may tend to zero arbitrarily slowly, in the following sense: for every positive nondecreasing sequence $\{a_n\}$, if $\sup_n a_n|C_n(E)|$ is finite for every E then $\sup_n a_n$ is finite. (Problem 4749 [1958, 453], treats the case $a_n=n$.)

Solution by B. J. Pettis, University of North Carolina. Let Σ be a σ-ring of sets, $\{C_n\}$ a sequence of complex countably additive functions on Σ, and $\{a_n\}$ a complex sequence. For each n, set $D_n=a_nC_n$ and let c_n and d_n be the total variations on Σ of C_n and D_n respectively, so that $d_n=|a_n|c_n$. If $\sup_n |D_n(E)|<\infty$ is true for each E in Σ, then by a theorem of Nikodym's (Dunford-Schwartz, *Linear Operators*, p. 309) the supremum K of the set $[|D_n(E)| : n\in\omega, E\in\Sigma]$ is finite. Clearly $4K\geqq d_n$ for every n, so that the l.u.b. δ of $\{d_n\}$ is finite and $\delta\geqq|a_n|c_n$ for every n. If also the lower limit γ of $\{c_n\}$ is positive then $\limsup|a_n|\leqq\delta/\gamma$, implying that $\{a_n\}$ is bounded. Since in the problem above $c_n=\int_0^1|e^{2\pi inx}|\,dx=1$, the desired conclusion is clear.

Also solved by the proposer.

Determinants Constructed from a Given Set of Elements

4902 [1960, 382]. *Proposed by R. A. Melter, University of Missouri*

At most how many different values may an nth order determinant have, if its elements are a given set of n^2 different, nonzero real numbers?

Solution by P. L. Renz, University of Pennsylvania, and J. M. Gorfinkel, Stanford University. There are $(n^2)!$ distinct $n\times n$ matrices which may be formed from n^2 algebraically independent elements. We divide the matrices into equivalence classes of matrices having the same determinant. Each such equivalence class has at least $(n!)^2$ elements as there is a set T of $n!^2$ distinct transformations

obtained by combinations of row and column permutation and transposition leaving the determinant of any $n \times n$ matrix invariant. On the other hand, the elements being algebraically independent, the value of $|(\theta_{ij})|$ completely determines the elements lying in the same row and column as θ_{ij} as those elements which do not multiply θ_{ij} in $|(\theta_{ij})|$. If $|(\theta_{ij})| = |(\phi_{ij})|$ for two of our $n!^2$ matrices, then we may bring the element θ_{11} to the 1, 1 place by transforming (ϕ_{ij}) by an element of T. Then, at worst by a transposition, we bring the elements of the first row and column of (θ_{ij}) into the first row and column respectively of the transformed (ϕ_{ij}) matrix. Continuing in this way, we transform (ϕ_{ij}) into $(\bar{\phi}_{ij})$ by elements of T until its first row and column, with the possible exception of the last two elements of the first column, are placed exactly as those of (θ_{ij}). Then, since $|\bar{\phi}_{ij})| = |(\theta_{ij})|$, it follows by the above that $(\bar{\phi}_{ij}) = (\theta_{ij})$ and thus that (ϕ_{ij}) is one of the $n!^2$ matrices obtained from (θ_{ij}) by transformations of T. Then the number of distinct values that the determinants of our matrices have is the number of distinct matrices divided by the number of matrices in each equivalence class, *i.e.*, $(n^2)!/n!^2$.

Also solved by Roberto Frucht and Michael Goldberg.

A Limit Problem

4905 [1960, 479]. *Proposed by Basil Gordon, Redstone Arsenal, Alabama*

Does the following limit exist:

$$\lim_{x \to 1^-} (1 - x)^{1/2} \prod_{n=0}^{\infty} (1 + x^{t^n})?$$

I. *Solution by Paul Erdös, University of Melbourne, Australia.* The following more general theorem holds: Let $a_1 < a_2 < \cdots$ be a sequence of integers and let $0 < \alpha < 1$. Then

$$\lim_{x \to 1} (1 - x)^{\alpha} \prod_{k=1}^{\infty} (1 + x^{a_k}) = c \tag{1}$$

is impossible for $c \neq 0$ and $c \neq \infty$. (This will imply a negative answer to the proposed problem since a simple calculation shows that the given limit cannot be 0 or ∞.)

To prove the theorem, assume that (1) is true for $c \neq 0, \infty$. Then

$$\limsup a_k/(a_1 + \cdots + a_{k-1}) > 1. \tag{2}$$

Assume that (2) is already proved. Write $\prod_{k=1}^{\infty} (1 + x^{a_k}) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$\lim_{x \to 1} (1 - x)^{\alpha} \sum_{n=0}^{\infty} b_n x^n = c, \qquad (c \neq 0, c \neq \infty).$$

But then by a theorem of Hardy and Littlewood (Proc. London Math. Soc., 1914, p. 174),

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_k/n^\alpha = c', \qquad (c' \neq 0, c' \neq \infty). \tag{3}$$

(In fact, $c'=c/\Gamma(1+\alpha)$.) But (3) is impossible since by (2) there is a fixed $\epsilon>0$ and arbitrarily large values of $m(=a_1+a_2+\cdots+a_{k_i})$ so that for every $m<n<m(1+\epsilon)$, we have $b=0$, which contradicts (3). Thus to complete the proof we have only to prove (2).

Assume (2) false. Then a simple argument shows lim sup $a_k^{1/k}\leqq 2$ (for if not, then for infinitely many k's, $a_k>(2+\epsilon)^{k-l}a_{k-l}$ for all $l<k$, which implies (2)). But lim sup $a_k^{1/k}\leqq 2$ implies that for every ϵ,

$$\lim_{x\to 1} (1-x)^{1-\epsilon} \prod_{k=1}^{\infty} (1+x^{a_k}) = \infty \tag{4}$$

which contradicts (1); thus our statement is proved. ((4) is evident: put $x=1-1/n$, $1+(1-1/n)^{a_k}=2+o(1)$ if $a_k=o(n)$ and the number of $a_k<m$ exceeds $(1+o(1))\log m/\log 2$ as $m\to\infty$.)

II. *Solution by J. H. Roberts, Duke University.* The limit does not exist. Letting f denote the given function, we have, for $0<x<1$,

$$f(x)f(x^2) = (1+x)^{1/2}, \tag{i}$$

$$\frac{f(x)}{f(x^4)} = \left(\frac{1+x}{1+x^2}\right)^{1/2}. \tag{ii}$$

Formula (ii) follows directly from the definition of f, formula (i) follows from the identity:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \prod_{n=0}^{\infty} (1+x^{2^n}), \qquad |x| < 1.$$

Now by (i), if $\lim_{x\to 1-} f(x)=L$ exists, then $L^2=2^{1/2}$ and $L=2^{1/4}$.

On the other hand, computation shows that if $\log_{10}(d) = -.00001$, $(d=.9999769744)$, then $\log_{10} f(d)>.076>(\log_{10} 2)/4$, whence $f(d)>2^{1/4}$. Now by (ii), $0<x<1$ implies $f(x^{1/4})>f(x)$. Thus, for the sequence $x_1=d$, $x_2=d^{1/4}$, $\cdots$, $x_{n+1}=x_n^{1/4}$, $\cdots$, we have $2^{1/4}<f(x_1)<f(x_2)<\cdots<f(x_n)<\cdots$, with all $x_n<1$. Thus $2^{1/4}$ cannot be the limit. Therefore the limit does not exist.

Also solved by Y. S. Chow, Sigvard Jacobs, Glenn M. Roe, and the proposer.

Editorial Note. However, $f(x)$ is bounded on $0\leqq x<1$. From (ii) we have $f(x)>f(x^4)>\cdots>f(x^{4^n})>\cdots$. But $x^{4^n}\to 0$ and $f(0)=1$. Thus $f(x)>1$ which also, in light of (i) proves $f(x)<\sqrt{(1+x)}$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Matrices. By W. V. Parker and J. C. Eaves. Ronald Press, New York, 1960. viii+195 pp. $7.50.

The first chapter of the book is a short discussion of preliminary ideas, mainly fields. The material following includes elementary properties of matrices and partitioning, singular and nonsingular matrices, equivalence of bilinear forms and matrices, rank, systems of linear equations, congruence, quadratic forms, vector spaces, determinants, invariant factors, characteristic vectors and equations, rational canonical form, minimum function, Hamilton-Cayley theorem, elementary divisors, Jordan normal form, and normal matrices. There are exercises following each section and just before the index an extensive bibliography appears.

This is an introductory text on matrices with abstraction kept to a minimum. It is clearly and carefully written with good detailed explanations and many examples to illustrate the theory. Matrices are introduced through linear forms rather than through linear transformations. In considering this book one should also be aware that a linear transformation is defined in it as a certain kind of matric equation rather than in the usual way as a linear mapping of a vector space. This latter concept is mentioned briefly in Chapter 8, but is not called a linear transformation, and the relations between matrices and linear transformations are not made clear.

HOWARD E. CAMPBELL
Michigan State University

An Introduction to Algebra for College Students. By W. A. Rutledge and Simon Green. Prentice Hall, Englewood Cliffs, N. J., 1960. ix+276 pp. $4.95.

This text is for college students who have had at most one year of high school algebra. It is, as the authors state in their preface, an attempt to utilize the postulational approach in the presentation of the algebra of the real number system.

The treatment of standard topics from Chapter 7 on is very good. The sections on exponents, radicals, functions, and graphing are carefully written.

In early chapters the field postulates are presented and an *attempt* is made to present the calculating techniques of algebra as theorems that may be derived from these postulates. A superficial reading of these first chapters leaves one with the impression that the authors' objectives have been reasonably well achieved, but close inspection discloses several shortcomings. The teacher who recognizes these weaknesses should be able to use the text effectively.

In general, the student is not taken into full partnership in the authors' program of deriving algebraic techniques from the postulates. For example, addition of integers is illustrated by three proofs of special cases (i) $6+(-3)=3$; (ii) $6+(-9)=-3$; (iii) $-4+(-3)=-7$. Then the old rules for addition of integers are stated. It seems quite probable that students will ignore the proofs and cleave to the rules.

Properties of numbers are postulated or tacitly assumed and then later on are proved in exercises. For example after several pages of working with *the* reciprocal and *the* negative of a real number it is proved, in an exercise, that each real number has a *unique* negative and each nonzero real number a *unique* reciprocal.

Subtraction is defined by the agreement that $a-b=a+(-b)$. However the important property that $b+x=a\rightarrow x=a-b$ is not proved. There is no proof that $a(b-c)=ab-ac$. An argument that $-(a-1)$ is the number $(-a)+1$ is clumsily presented with no justification for several of the steps in the argument. One might almost as well instruct students to "take off the parentheses and change signs."

On page 15 subtraction occurs before the operation has been defined. Much of the language is objectionable. One finds the familiar confusion of symbol with concept. Examples of careless language are: "The symbol $-a$ might or might not be a negative number." "The negative of a number need not have the negative symbol." "3/1 is an integer; 6/2 is not an integer by form but has a value which is an integer." "When we express a number with the usual arithmetic symbols, such as 34, we shall call it an *arithmetic* number or call this the *value* of the number." "We define an *exponent n* of the base a · · · . We are going to define the nth root of the number a, if there is one, by the symbol $a^{1/n}$. The fraction $1/n$ serves to define an nth root." "A solution is a known number that will make the equation true."

Although we feel strongly that the text should be carefully rewritten, cleaning up the language and reorienting the axiomatic development, yet in its present form it is certainly superior to the usual cookbook text. In the hands of a competent instructor it could be used as the basis of a satisfactory course.

CHARLES BRUMFIEL
The University of Michigan

The Modern Aspects of Mathematics. By Lucienne Felix. Translated by Julius H. Hlavaty and Fancille H. Hlavaty. Basic Books, New York, 1960. xiii+194 pp. $5.00.

This is a book about mathematics which contains much mathematics. It sets the stage for the advent of the works of the Bourbakists. It describes their attempts to broaden the foundations of logic and mathematics sufficiently to support present-day research and to provide a basis for a unified structure in which all branches of mathematics, both old and new, intuitive and axiomatic,

can find their place and be properly evaluated. There is a summary of modern logic, set theory, modern algebra, and topology. This is a well-written book, spiced with interesting historical items and examples of mathematical theory. It concludes with some remarks on the implications of the concepts described on the pre-college curriculum, and the reader must conclude with the author that a broad training in mathematics is a necessity, at least for the teachers of bright students, both at the secondary and elementary levels. Mathematics teachers at all levels should find this book full of interesting material about their profession.

C. L. SEEBECK, JR.
University of Alabama

Foundations of Modern Analysis (vol. X of *Pure and Applied Mathematics*). By J. Dieudonné. Academic Press, New York, 1960. xiv+361 pp. $8.50.

This unusual, in part revolutionary, book covers parts of analysis which the author believes should be known to all graduate students in mathematics. Chapters I through VII are familiar enough: elements of set theory; real numbers; metric spaces (elementary theory, not including Baire category, but including the Stone-Weierstrass theorem and the Tietze-Urysohn extension theorem for metric spaces); normed spaces; Hilbert spaces; spaces of continuous functions (the influence of Bourbaki is discernible in the author's discussion of "regulated" functions). Chapters VIII and IX, titled *Differential Calculus and Analytic Functions*, respectively, are radical indeed. Derivatives are defined for continuous mappings of a Banach space E into a Banach space F; the derivative when it exists is a bounded linear mapping of E into F. The mean value theorem, primitive functions, Jacobians, higher derivatives all appear, with logically precise definitions, but in disguises that may prove highly effective. The Lebesgue integral is not discussed. Analytic functions are also treated with maximum generality: usually they are functions from open subsets of n-dimensional real or complex space to a real or complex Banach space that admit local power series expansions. A weak (homotopic) form of Cauchy's theorem is established. Chapter X, titled *Existence Theorems*, is also quite abstract. Chapter XI, *Elementary Spectral Theory*, is a standard treatment.

The book is written with great precision, and is provided with a huge number of problems. (Some of these, the great Picard theorem for example, are highly sophisticated.)

The author's style is vigorous, and his views unambiguous. In discussing many traditional topics, his attitude seems to be, never call a spade a spade if you can call it a damned old shovel. For students well grounded in classical analysis, study of this book will undoubtedly be a stimulating and exciting adventure. Whether it will become a standard text for first-year real variables courses, only time will tell.

EDWIN HEWITT
University of Washington

Elementary Mathematical Programming. By Robert W. Metzger. Wiley, New York, 1958. ix+246 pp. $5.95.

This book describes in simple terms some methods used in solving what is generally known as the "linear programming problem," *i.e.*, the problem of finding a vector which minimizes a linear form, subject to certain linear constraints. The author has endeavored to make the concepts and techniques of linear programming accessible to those having little training in mathematics, and in this endeavor he has succeeded rather well. The treatment of mathematical material such as this in a nonmathematical way seems inevitably to require at least the appearance of prolixity, if not the actuality, so that patient, careful reading is necessary for a full understanding of the subjects considered.

After a brief introductory chapter dealing with the history of Operations Research and the areas in which mathematical programming has been found useful, the author in Chapter 2, "Distribution Methods," takes up the so-called "transportation problem" in which, in its simplest form, it is required to ship various amounts of a product from each of m warehouses to satisfy the demands of each of n customers in such a way as to minimize the shipping costs. The "stepping-stone," modified distribution, and Vogel approximation methods of solving this problem are presented in detail. The text here is very amply supplemented by figures. The reader with very little more than a minimum of mathematical preparation should find the mathematical formulation of the three-dimensional problem at the end of this chapter satisfying. Chapter 3 takes up the simplex method of solution of the linear programming problem. The method is illustrated by means of the simple, two-product, manufacturing situation. The introduction here leaves something to be desired and the only hint as to the origin of the term *simplex* is misleading in that it implies that it may derive from the nature of the method, an idea of which the reader, admittedly with the help of an admonition by the author, is shortly disabused. The method is well illustrated and the rules for its use listed. The geometric interpretation of the problem is then given. It would seem that a clearer idea of the problem might be imparted were this interpretation presented at the beginning. The dual problem and the modified simplex method are then treated. Chapter 4 is devoted to approximation methods; Chapter 5 to two applications based on real situations. Chapter 6 (six pages) points out the availability of digital computer programs, specifically for IBM computers, for the solution of larger problems, but is rather misleading to those interested in using the IBM 650 for their problems but who are unfamiliar with its capabilities and those of auxiliary machines invariably found in such a computing center. (One rarely is reduced to interpreting punched-card output manually, and the one and one-half hours mentioned is total time for *all* phases of the processing of the problem.) Chapters 7–10 present in clear fashion problems in production planning, stock splitting, material-handling scheduling, and job and salary evaluation.

There are a few minor defects in exposition and technique, *e.g.*, occasional

failure to italicize technical terms when first mentioned, and at least one definition of the "is when" type, but in general it is a well-written book. There are no exercises.

FRANKLIN S. MCFEELY
Montana State College

A Course in Applied Mathematics. By Derek F. Lawden. The English Universities Press, London, 1960. xv+655 pp. 70/–. (about $7.00).

This excellent work is essentially an introduction to classical mathematical physics. The major subdivisions (together with the pages devoted to each) are as follows: Dynamics (270), Statics (178), Field Theory (193)—mostly electricity and magnetism, and Hydromechanics (87). The minimum mathematical requirement for progress through the book at a satisfactory rate is a good knowledge of advanced calculus, including some vector analysis exclusive of the gradient, divergence, and curl, for as asserted in the preface: "It is also assumed that the reader will be attending a course of lectures in pure mathematics during which he will study the theory of linear differential equations both ordinary and partial, the definition and evaluation of surface and volume integrals and the notation and techniques of vector analysis."

According to a statement on the jacket: "Every effort has been made to assist those students who must depend largely on their own resources while studying the subject." In the reviewer's opinion, the author has succeeded admirably in fulfilling this purpose, and this special suitability for unassisted study is perhaps the cardinal characteristic of the book. Altogether, this work presents many instances of superior expository writing. The rigor level is a cut above the average for textbooks in this field—a circumstance that is conducive to clarity; and the theoretical developments are sufficiently detailed for solo study. The chapters are organized into short integrated units consisting of either a single article or a set of two or three closely related articles followed in either case by illustrative examples. This structure should be welcomed by those readers who by choice or necessity will use short study periods; perhaps in addition it will serve as a deterrent to mental indigestion. The articles are usually of two or three pages each and the illustrative problems are worked out in detail. There are 546 unsolved problems.

For the mathematician interested in theoretical physics this book might serve as a source for quick references at the introductory level. There are a few places where one might wish to introduce a mean-value theorem or some other device for improving the rigor but on the whole the mathematical developments are satisfactory and in some cases, as in the treatment of fields in dielectric and magnetic media, the strategies employed are assuredly time-saving. In conclusion, this work is a valuable addition to the expository literature on mathematical physics.

HOMER V. CRAIG
University of Texas and Boeing Airplane Co.

Calculus. By Wray G. Brady and Maynard J. Mansfield. Little, Brown, Boston, 1960. ix+456 pp. $8.00.

This book "offers a course in modern mathematics via the calculus, and on the other hand, a course in calculus via modern mathematics."

In the first chapter elementary notions of set theory, real inequalities, intervals, bounds, least upper bounds, etc. are introduced. In this discussion the field properties of the real numbers are assumed. In the second chapter, functions (as sets of ordered pairs), limits, and continuity are defined with great care and yet with excellent motivational comments. Proofs usually omitted in elementary books are given for many statements (*e.g.* the boundedness theorem for continuous functions). Throughout the book a strictly up-to-date notation and viewpoint is maintained.

Except for the unusually rigorous exposition and the liberal use of functional equations, chapters III through IX cover the standard material on differentiation and anti-differentiation of real functions of one variable with applications.

Chapter X contains a radical departure from the orthodox elementary treatment of integration. The Riemann-Stieltjes integral is introduced and it is shown that averages, for countable collections, can be expressed as integrals. This is done with the following machinery. The concept set function and measure are introduced with a finite measure and a length measure given as examples. Certain measurable functions, called "traits," (*e.g.*, random variables), and then a generalized concept of distribution function are introduced. Finally infinite averages (of traits) are defined and relations with definite integrals shown. There follows a discussion of standard material on integration with benefit of the more general treatment afforded by use of the Stieltjes integral at the outset. In particular the authors are able correctly to discuss probability distributions in Chapter XIV.

As the above sketch should indicate, the book is a far cry from the material and point of view assumed in most beginning calculus courses, and would be too strong a diet for most freshmen or sophomores. It would, however, be eminently suitable for an honors course for mathematics and science majors.

The book is clearly written (although the introductory material on measure, traits, sums, etc. in Chapter X could profit from more examples and better diagrams), and appears to be free of typographical errors. The authors have written an excellent book with extreme care.

R. J. Nelson
Case Institute of Technology

BRIEF MENTION

Modern Computing Methods, Notes on Applied Science No. 16. National Physical Laboratory. Purchasable through Her Majesty's Stationery Office, 423 Oxford Street, London W 1, England, 1959. vi+129 pp. $1.52.

An excellent introduction to computer-oriented algebraic mathematics and finite

differences, including differential equations. This is apparently the same as the book published in hard-back cover by the Philosophical Library of New York at $8.75.

Progress in Automation, Vol. I. Edited by Andrew D. Booth. Academic Press, New York, 1960. viii+231 pp. $8.50.

Another book from our British cousins, this time devoted mainly to industrial use of computers in control situations.

Advances in Computers, Vol. I. Edited by Franz L. Alt. Academic Press, New York, 1960. 316 pp. $10.00.

This collection of six papers includes General Purpose Programming for Business Applications by C. C. Gotlieb; Numerical Weather Prediction by N. A. Phillips; The Present Status of Automation Translation of Languages by Y. Bar-Hillel; Programming Computers to Play Games by A. L. Samuel; Machine Recognition of Spoken Words by R. Fatehchand; and Binary Arithmetic by G. W. Reitwiesner.

Automatic Language Translation. By Anthony G. Oettinger. Harvard University Press 1960. 380 pp. $10.00.

No mere popular treatise filled with generalities, but a serious well-written account of the Russian translation project at the Harvard computation laboratory. It includes some good remarks on computer programming, particularly as it applies to data processing of languages.

Introduction Mathématique a la Mécanique des Fluides. By M. Caius Jacob. Gauthier-Villars, Paris, 1959. 1286 pp. $10.00.

A hefty tome dealing with both compressible and incompressible fluid mechanics.

Handbuch Der Schulmathematik. By Georg Wolff. Hermann Schroedel Verlag K G, Berlin HANNOVER, Darmstadt Postfach 89, 1960, 296 pp.

Algebra for Commerce and Liberal Arts. By A. K. Bettinger and W. A. Dwyer. Pitman, New York, 1960. 243 pp. $4.00.

A nice job of printing of an extremely routine presentation.

Intermediate Algebra, Alternate Edition. By Lovincy J. Adams. Holt, Rinehart and Winston, New York, 1960. xiii+414 pp. $4.50.

Plane Trigonometry, 4th Edition. By Fred W. Sparks and Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+308 pp. $5.25. An old favorite.

Ingenious Mathematical Problems and Methods. By L. A. Graham. Dover, New York, 1960. vii+237 pp. $1.45.

There are some charming bits in this paperback collection. Originally published in the "Private Corner for Mathematicians" of the Graham DIAL.

An Introduction to Linear Programming and the Theory of Games. By S. Vajda. Wiley, New York, 1960. 76 pp. $2.25.

A brief but stimulating introduction to Vajda's better-known works.

Espaces Topologiques, Fonctions Multivoques. By Claude Berge. Dunod, Paris, 1959. 272 pp.

Lectures on Ergodic Theory. By Paul R. Halmos. Chelsea, N. Y., 1956. vii+99. $2.95.

A reprint of the Mathematical Society of Japan's 1956 edition of Halmos' 1955 lectures at the University of Chicago.

Engineering Mathematics. By J. Blakey and M. Hutton. Philosophical Library, New York, 1960. 603 pp. $10.00.

This seems to be a somewhat "cook-book" version of Blakey's *University Mathematics* with some additional work on statistics, relaxation methods and Laplace transformations.

The Real Projective Plane, 2nd Edition. By H. M. S. Coxeter. Cambridge University Press, New York, 1960. xi+226 pp. $3.75 (paperback).

Congratulations to Cambridge Press for bringing us a revised edition of Coxeter's excellent 1949 work. This is not merely a reprinting of the previous book. Many changes have been made.

The Theory of Linear Economic Models. By David Gale. McGraw-Hill, New York, 1960. 350 pp. $9.50.

Gale doesn't expect students of economics to know the necessary mathematics when they start his book, but neither does he undertake to teach linear theory without developing mathematics. He develops those portions of vector and matric theory which he will need, and those only—a commendable procedure.

Normed Rings. By M. A. Naimark, translated from the Russian by Leo F. Boron. P. Noordhoff-Groningen, Holland, 1960. xvi+560 pp. $13.00.

There is an amazing amount of information packed into less than 600 pages. The first 154 pages deal with basic ideas from topology and functional analysis, then in succession come: fundamental concepts in propositions in the theory of normed rings, commutative normed rings, representations of symmetric rings, special rings, group rings, rings of operators in Hilbert space, and decomposition of a ring of operators into irreducible rings.

Riemann Surfaces. By Lars V. Ahlfors and Leo Sario. Princeton University Press, Princeton, N. J., 1960. xi+382 pp. $10.00.

This excellent book will be reviewed in the Bulletin of the American Mathematical Society in the forthcoming issue.

Cartesian Geometry of the Plane. By E. M. Hartley. Cambridge University Press, New York, 1960. xi+324 pp. $3.75.

Frankly, I would enjoy teaching a course in analytic geometry from this text. If your school still gives a separate course in analytic geometry, I sincerely recommend that you obtain a copy of this book for consideration.

Grundlagen Der Analysis. By Edmund Landau. Chelsea, New York, 1960. 173 pp. $1.95.

With a complete German-English vocabulary, this old favorite, both of mathematicians and of students who wish to pass a German reading examination without knowing any German, is again available in a modern reprint. The publisher has even translated the preface to save the above students possible difficulty in translation.

The New Mathematics. By Irving Adler. Mentor, New York, 1960. 192 pp. 50¢.

Pick up a copy of this Mentor edition at your corner drugstore and look it over for yourself. It is worthwhile.

Éléments D'Algébre. By Gaston Julia. Gauthier-Villars, Paris, 1959. viii+207 pp. $7.93.

Applied Boolean Algebra. By Franz E. Hohn. Macmillan, New York, 1960. xx+139 pp. $2.50.

A fine paperback which you will wish to call to the attention of your electrical engineering students. Anyone unfamiliar with the use of Veitch diagrams will want to read Chapter 4. This valuable, practical tool is often overlooked.

Introduction A L'utilisation Pratique de la Transformation De Laplace. By Gustav Doetsch. Gauthier-Villars, Paris, 1959. viii+198 pp. $7.50.

Modern Elementary Statistics, 2nd Edition. By John E. Freund. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+413 pp. $6.95.

One of the better modern books on statistics for those who do not have a calculus background.

The Mathematical Foundations of Quantum Statistics. By A. Y. Khinchin, translated from the Russian by Irwin Shapiro. Graylock Press, Albany, N. Y., 1960. xi+231 pp. $10.00.

Thermodynamics and statics based on a sound statistical background.

Contributions to Probability and Statistics. Edited by Olin, Ghurye, Hoeffding, Madow and Mann. Stanford University Press, Stanford, California, 1960. x+517 pp. $6.50.

This series of essays in honor of Harold Hotelling contains 38 short papers by people who have been closely associated with Professor Hotelling. What more fitting tribute can there be to a great scholar?

Five welcome reprints from Dover Press, New York, 1960:

Algebras and Their Arithmetics. By Leonard Eugene Dickson. xii+241 pp. $1.35.

Statistics Manual. By Crow, Davis and Maxfield. xvii+288 pp. $1.55.

Differential Equations for Engineers. By Philip Franklin. vii+299 pp. $1.65.

The Applications of Elliptic Functions. By Alfred George Greenhill. xi+357 pp. $1.75.

Coordinate Geometry. By L. P. Eisenhart. xi+297 pp. $1.65.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. R. W. Bass, Mathematician and Chief Scientist for the American Manufacturing Corporation, Aeroscope Division, Catonsville, Maryland, has been named Maryland's Outstanding Young Scientist of 1960 by Maryland Academy of Sciences.

Harpur College: Associate Professor A. D. Ziebur, Ohio State University, has been appointed Associate Professor; Dr. K. W. Anderson, University of Illinois, and Dr. Lily H. Seshu, University of Toronto, have been appointed Assistant Professors.

Hunter College: Associate Professor I. H. Rose, University of Massachusetts, has been appointed Associate Professor; Assistant Professors A. D. Bradley and Carolyn Eisele have been promoted to Associate Professors; Professor L. S. Hill and Assistant Professor Helen K. Kutman retired September 1, 1960.

Pennsylvania State University: Mr. W. D. Bouwsma, University of Michigan, has been appointed Assistant Professor; Mr. L. F. Boron, St. Vincent's College, has been appointed Instructor; Associate Professors Raymond Ayoub and J. B. Bartoo have been promoted to Professors; Assistant Professors C. C. Faith and Mary L. McCammon have been promoted to Associate Professors; Associate Professor C. C. Faith is spending the academic year at the Institute for Advanced Study; Professor Orrin Frink is spending the academic year at University College, Dublin, Ireland.

Polytechnic Institute of Brooklyn: Dr. J. S. Lomont, International Business Machines, Yorktown Heights, New York, has been appointed Associate Professor; Assistant Professor George Bachman, Rutgers University, has been appointed Assistant Professor; Mr. Clifford Marshall, Miss Sulbha Agarwal, Miss Helen Bowden, and Mrs. Ruth G. Favro have been appointed Instructors; Associate Professor J. C. Scanlon has been promoted to Professor; Mr. R. B. Lowe has been promoted to Assistant Professor.

Purdue University: Associate Professor Harley Flanders, University of California, Berkeley, has been appointed Professor; Associate Professor R. L. Blair, University of Oregon, Assistant Professor Melvin Carter, North Carolina State College, Assistant Professor L. J. Cote, Syracuse University, Dr. R. A. Gambill, General Motors, Indianapolis, Indiana, Dr. J. H. Michael, University of Adelaide, and Assistant Professor G. J. Rieger, University of Maryland, have been appointed Associate Professors; Assistant Professor A. R. Amir-Moez, Queens College, Dr. R. B. Kane and Assistant Professor Rosemarie Stemmler, University of Illinois, and Dr. J. C. Lillo, RIAS, have been appointed Assistant Professors; Mr. E. H. Lehman, North Carolina State College, Mr. L. F. Bruening, Mr. R. E. Hughs, and Mr. Kenzo Seo have been appointed Instructors; Associate Professors Philip Dwinger, Melvin Henriksen, Meyer Jerison, and Stanley Reiter have been promoted to Professors; Assistant Professors A. H. Copeland, Jr., Morris Skibinsky, R. F. Williams and R. E. Zink have been promoted to Associate Professors; Drs. James Hogg, E. M. McNally, L. D. Pyle, and R. F. Scott, have been promoted to Assistant Professors.

Rutgers, The State University: Drs. A. W. Adler, Massachusetts Institute of Technology, R. W. Carroll, University of Nancy, France, B. I. Gross, University of Pennsylvania, Benjamin Muckenhoupt, De Paul University, and Samuel Park, University of Pittsburg, have been appointed Assistant Professors; Mr. R. T. Bumby and Mr. F. M. Sand, Princeton University, Mr. J. H. Oppenheim, University of Illinois, Dr. D. R. Ostberg, University of California, Berkeley, and Mr. Thomas Paley, Skidmore College, have been appointed Instructors; Associate Professors K. G. Wolfson, Joshua Barlaz, and H. J. Zimmerberg have been promoted to Professors; Assistant Professor J. C. E. Dekker has been promoted to Associate Professor; Mr. R. E. Bryan and Mr. R. C. Courter have been promoted to Assistant Professors; Professor M. G. Galbraith retired with the title of Professor Emeritus.

San Diego State College: Mr. E. I. Deaton, University of Texas, Mr. W. E. de Maligon, State University of South Dakota, Dr. Robert DeZur, University of Wyoming, Mr. L. D. Fountain, University of Nebraska, Mr. R. L. Van de Wetering, Stanford University, Mr. Herbert Gindler, and Mr. Raymond Killgrove have been appointed Assistant Professors; Miss Elizabeth Otten has been appointed Lecturer; Mr. Max Bergstrom, Mr. Rolando Peinado, and Miss Shirley Weihe have been appointed Instructors; Associate Professors A. R. Harvey and L. G. Riggs have been promoted to Professors; Assistant Professors Dean Branstetter, Peter Shaw, N. B. Smith, Le Roy Warren, and Margaret Willerding have been promoted to Associate Professors.

University of California, Berkeley: Professors J. L. Koszul and René Thom, University

of Strasbourg, France, Alex Rosenberg and Daniel Zelinsky, Northwestern University, Wolfgang Rothstein, University of Münster, Germany, and Jerome Sachs, Chicago State Teachers College, have been appointed Visiting Professors; Dr. Stephen Smale, Institute for Advanced Study, has been appointed Associate Professor; Drs. G. E. Bredon and M. W. Hirsch, Institute for Advanced Study, Chen-Chung Chang, University of California, Los Angeles, Adam Koranyi, Harvard University, D. S. Scott, University of Chicago, and Hirofumi Uzawa, Stanford University, have been appointed Assistant Professors; Dr. Steven Bryant, Fresno State College, Dr. L. E. Dubins, University of California, Berkeley, Assistant Professor S. H. Dwivedi, N.R.E.C. College, Khurja, India, Dr. Linda Naim, Maitre de Conferences, Institut Fourier, Grenoble, France, Drs. Hugo Rossi and D. F. Wehn, Princeton University, and Assistant Professor J. H. Wells, University of North Carolina, have been appointed Visiting Assistant Professors; Drs. Gertrude I. Heller, RIAS, and F. J. Kosier, Michigan State University, have been appointed Instructors; Drs. R. H. Abraham, University of Michigan, A. T. Lundell, Brown University, and R. R. Phelps, Institute for Advanced Study, have been appointed Lecturer and Assistant Research Mathematicians; Mr. J. M. Cook, Argonne National Laboratory, has been appointed Lecturer; Mr. P. L. Cavailles, Paris, France, has been appointed Acting Instructor; Associate Professor R. J. de Vogelaere has been promoted to Professor; Assistant Professors Jacob Feldman, P. E. Thomas, and R. L. Vaught, have been promoted to Associate Professors.

University of Cincinnati: Professor I. A. Barnett has been appointed Acting Head of the Department of Mathematics; Associate Professor Charles Saltzer, Case Institute of Technology, has been appointed Professor; Dr. Lee Suyemoto has been appointed Acting Assistant Professor; Mr. R. M. Lotspeich, Oklahoma State Uuiversity, Mr. R. H. Rolwing, Christian Brothers College, Mr. M. L. Brown, Mr. J. R. Downing, Mr. A. L. Haste, and Mr. E. V. Martin have been appointed Instructors.

University of Florida: Dr. J. E. Maxfield, Naval Ordnance Test Station, China Lake, California, has been appointed Professor and Head of the Department of Mathematics; Professor Paul Civin, University of Oregon, has been appointed Visiting Research Professor; Assistant Professor Henryk Minc, University of British Columbia, has been appointed Associate Professor; Mr. Del Willard, Baldwin-Wallace College, Mrs. Jane M. Day, and Mr. J. L. Tilley have been appointed Instructors; Mr. N. W. Hill, Radio Corporation of America at Patrick Air Force Base, Florida, and Mrs. Petee S. Jung, University of Massachusetts, have been appointed Interim Instructors; Professor F. W. Kokomoor retired as Head of the Department of Mathematics on June 30, 1960; Professor C. G. Phipps retired June 30, 1960.

University of Illinois: Assistant Professor S. I. Goldberg, Wayne State University, has been appointed Visiting Associate Professor; Acting Assistant Professor A. Feinstein, Stanford University, has been appointed Assistant Professor; Dr. J. L. Britton, Glasgow University, Scotland, and Assistant Professor E. E. Kohlbecker, University of Utah, have been appointed Visiting Assistant Professors; Mr. M. J. Wicks, University of Malaya, Singapore, Malaya, has been appointed Visiting Instructor; Drs. Betty Detwiler, University of Kentucky, Leone Low, Oklahoma State University, David Sachs, Illinois Institute of Technology, P. M. Weichsel, California Institute of Technology, and E. C. Weinberg, Purdue University, have been appointed Instructors; Associate Professors W. W. Boone and Alex Heller have been promoted to Professors; Assistant Professors D. L. Burkholder, R. C. Langebartel, Echo D. Pepper, L. A. Rubel and R. A. Wijsman have been promoted to Associate Professors; Professor J. L. Doob has been made a charter member of the Center of Advanced Study and Professor W. W. Boone is an associate member for this year.

University of Kansas: Professor S. M. Shah, Muslim University, Aligarh, India, has been appointed Visiting Professor; Dr. W. C. Nemitz, Ohio State University, has been appointed Assistant Professor; Dr. Andrew Page, Kings College, has been appointed

Visiting Instructor; Mrs. Carol H. Bassett has been appointed Instructor; Assistant Professor A. H. Kruse has been promoted to Associate Professor; Associate Professor A. H. Kruse is on leave during the academic year of 1960–61 as Research Professor at the New Mexico State University Research Center; Professor R. M. Schatten is on leave for the academic year of 1960–61 as Visiting Professor at the University of California, Los Angeles; Associate Professor Florence Black retired June, 1960, with the title of Associate Professor Emeritus.

University of Massachusetts: Assistant Professor M. D. Barr, Eastern Michigan University, Mr. J. R. Brown and Miss Eleanor Killam, Yale University, Dr. R. A. McHaffey, Rutgers University, Mr. Torsten Norvig, Brown University, and Mr. C. A. Riley, University of Michigan, have been appointed Assistant Professors; Mr. W. J. Halm, University of Kansas, and Mr. R. K. Mento have been appointed Instructors; Assistant Professors R. R. Archer and D. J. Dickinson have been promoted to Associate Professors; Miss Lorraine D. Lavallee has been promoted to Assistant Professor; Associate Professor A. G. Azpeitia is on leave during the academic year of 1960–61 at Brown University.

University of Miami: Professor Emeritus Tomlinson Fort, University of Georgia, Dr. B. E. Howard, University of Chicago, and Professor Andrew Sobczyk, University of Florida, have been appointed Professors; Miss Jacqueline E. H. Elliott and Mr. R. L. Kelley have been appointed Instructors; Dr. E. F. Low, Jr., has returned from a year's leave of absence at the Institute of Mathematical Sciences, New York University, and has been promoted to Associate Professor.

University of Utah: Assistant Professors W. J. Coles, E. A. Davis, E. E. Kohlbecker and D. V. V. Wend have been promoted to Associate Professors; Associate Professor E. E. Kohlbecker is on leave for the academic year of 1960–61 at the University of Illinois; Associate Professor J. H. Barrett has returned from leave with the Mathematics Research Center, University of Wisconsin; Dr. Lida K. Barrett has returned from leave at the University of Wisconsin.

University of Virginia: Assistant Professors Marvin Rosenblum, P. E. Conner, and E. C. Paige, have been promoted to Associate Professors; Dr. N. F. G. Martin has been promoted to Assistant Professor.

Washington State University: Professor T. G. Ostrom, Montana State University, has been appointed Professor; Messrs. L. D. Coffin, J. A. Johnson, E. E. Mayer, and R. McCormmach have been appointed Instructors.

West Virginia University: Professor J. K. Stewart has been appointed Chairman of the Department of Mathematics; Miss Jean Loudin, Otterbein College, and Mrs. Judith Hall, West Virginia University, have been appointed Instructors; Associate Professor A. B. Cunningham has been promoted to Professor; Dr. H. A. Davis, retired in September, 1960, as Chairman of the Department of Mathematics with the rank of Professor.

Mr. C. B. Baytop, Albama Agricultural and Mechanical College, has been appointed Instructor at Howard University.

Mr. J. A. Brown, University of Delaware, has accepted a position as Operations Analyst with the Johns Jopkins Research Office, Bethesda, Maryland.

Professor Lamberto Cesari, Purdue University, has been appointed Professor at the University of Michigan.

Assistant Professor H. L. Crowson, University of Florida, has accepted a position as Staff Mathematician with International Business Machines, Bethesda, Maryland.

Associate Professor John Dyer-Bennet, Purdue University, has been appointed Associate Professor at Carleton College.

Associate Professor Virginia I. Felder, Mississippi Southern College, has been promoted to Professor.

Associate Professor Leonard Gillman, Purdue University, has been appointed Chairman of the Department of Mathematics at the University of Rochester.

Mr. W. R. Hydeman, Touche, Niven, Biley and Smart, Detroit, Michigan, has accepted a position as Manager of the Administration Systems Planning Division of Lockheed Missiles and Space Division, Lockheed Aircraft Corporation, Sunnyvale, California.

Dr. Ronald Jacobowitz, Princeton University, has been appointed Instructor at the Massachusetts Institute of Technology.

Mr. L. A. Kenna, Radio Corporation of America, Tucson, Arizona, has been appointed Instructor at the University of Arizona.

Assistant Professor H. T. La Borde, University of Cincinnati, has been appointed Associate Professor at the University of South Carolina.

Mr. W. R. LaCour, Northwestern State College, has accepted a position as Mathematician with the United States Department of Agriculture, Dallas, Texas.

Professor L. C. Lay, Pasadena City College, has been appointed Professor at Orange County State College.

Mr. Lowell Leake, Jr., University of Wisconsin, has been appointed Instructor at the University of Cincinnati.

Associate Professor R. C. Meacham, University of Florida, has been appointed Head of the Department of Mathematics at Florida Presbyterian College.

Professor H. M. MacNeille, Washington University, has been appointed Professor and Head of the Department of Mathematics at Case Institute of Technology.

Mr. H. H. Manker, Kansas State College, has accepted a position as Programmer with International Business Machines, Endicott, New York.

Associate Professor G. H. Miller, Western Illinois University, has been appointed Professor at State Teachers College at Towson, Maryland.

Mr. N. L. J. Oldani, University of Detroit, has accepted a position as Analyst with A. C. Spark Plug Company, Oak Creek, Wisconsin.

Mr. Wade Petersen, Washington University, has been appointed Instructor at St. Louis University.

Assistant Professor R. J. Pitts, Los Angeles State College, has been promoted to Associate Professor.

Professor Hans Samelson, University of Michigan, has been appointed Professor at Stanford University.

Mr. R. J. Schwabauer, University of Nebraska, has been appointed Instructor at the State University of South Dakota.

Mr. C. E. Seabold, Standard Oil Company of Ohio, Cleveland, Ohio, has accepted a position as Programmer with Applied Data Systems, Princeton, New Jersey.

Professor O. T. Shannon, Agricultural, Mechanical and Normal College, Pine Bluff, Arkansas, has been appointed Chairman of the Department of Mathematics and Physics.

Dr. Oved Shisha, Harvard University, has accepted a position as Mathematician with the Numerical Analysis Section of the National Bureau of Standards, Washington, D. C.

Mr. E. A. Stavinoha, Douglas Aircraft Company, Tulsa, Oklahoma, has accepted a position as Mathematician in the Research Laboratory of the Research and Development Operations, ABMA, Redstone Arsenal, Alabama.

Dr. Beauregard Stubblefield, University of Michigan, has accepted a position as Programming Specialist with International Electric, Paramus, New Jersey.

Mr. G. L. Sward, Cornell University, has been appointed Instructor at Vassar College.

Mr. John Weissman, Rutgers University, has accepted a position as Senior Engineer-Research with Autonetics, Downey, California.

Mr. J. P. Williams, University of Michigan, has accepted a position as Engineer with Sperry Gyroscope, Syosset, Long Island, New York.

Professor E. T. Bell, California Institute of Technology, died December 21, 1960. He was a Charter Member of the Association and former President of the Association 1931–32.

Rev. J. G. Burke, Mount St. Mary College, died September 6, 1960. He was a member of the Association for 36 years.

Professor C. T. Bumer, Clark University, died March 14, 1960. He was a member of the Association for 36 years.

Mr. Morris Dansky, Creighton University, died November 7, 1960. He was a member of the Association for 11 years.

Mr. J. J. Dodd, Wisconsin State College, died April 3, 1960. He was a member of the Association for 11 years.

Professor R. L. Erickson, retired from Lakeland College, died November 21, 1960. He was a member of the Association for 13 years.

Professor Alfred Errera, L'Université Libre de Bruxelles, Belgium, died September 18, 1960. He was a member of the Association for 21 years.

Associate Professor W. L. Fields, Hampton Institute, died May 29, 1960. He was a member of the Association for 19 years.

Professor Emeritus Edward Fleisher, Brooklyn College, died July 8, 1960. He was a member of the Association for 34 years.

Professor Emeritus C. H. Forsyth, Dartmouth College, died November 2, 1960. He was a Charter Member of the Association.

Professor Emeritus Isabel Harris, University of Richmond, died October 21, 1960. She was a member of the Association for 36 years.

Associate Professor B. M. Ingersoll, Arizona State University, died October 3, 1960. He was a member of the Association for 16 years.

Professor H. H. Irwin, Washington State University, died June 9, 1960. He was a member of the Association for 14 years.

Associate Professor G. B. Lang, University of Florida, died July 21, 1960. He was a member of the Association for 25 years.

Mr. A. L. Milner, University of Alabama, died June 30, 1960.

Associate Professor J. W. Popow, United States Naval Academy, died November 28, 1960. He was a member of the Association for 13 years.

Sister Mary Clementia, S.S.F., St. Mary's Academy, New Orleans, Louisiana, died March 7, 1960.

Associate Professor Hidehiko Yamabe, Northwestern University, died November 20, 1960. He was a member of the Association for 5 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The sixth annual meeting of the Northeastern Section of the Mathematical Association of America was held at Wesleyan University, Middletown, Connecticut on November 26, 1960. Professor J. G. Kemeny, Chairman of the Section, presided at the morning session and Professor D. E. Christie presided at the afternoon session. There were 166 people registered, including 113 members of the Association.

Officers chosen for 1960–61 were Professor D. E. Christie, Bowdoin College, Chairman; Professor R. A. Rosenbaum, Wesleyan University, Vice-Chairman; Mr. R. S.

Pieters, Phillips Academy, Secretary-Treasurer. Written notice of the following amendments to the By-Laws of the Section, as recommended by the Executive Committee, was sent to all members of the Section on October 25, 1960. These were passed by unanimous vote of those members of the Association present at the meeting.

Article III, section 2. Add the words, "and the sectional governor of the Mathematical Association of America." This makes the sectional governor a member of the Executive Committee.

Add "Article V. Registration Fees. The Executive Committee may, at its discretion, charge a registration fee of not more than $1.00 at the Annual Meeting. These fees shall be used to help pay the expenses of conducting the business of the Section."

By invitation of the Executive Committee the following talks were presented:

1. *An example in mathematical logic*, by Professor Hartley Rogers, Jr., Massachusetts Institute of Technology.

A decision procedure for the elementary theory of the ordering of the reals is described (Langford 1927). Its formulation is used to suggest some of the main traditional and present concerns of mathematical logic. The distinction between semantical and syntactical concepts is emphasized.

2. *The transfinite diameter*, by Professor Einar Hille, Yale University.

The transfinite diameter (Fekete, 1923) is a function of closed bounded sets in the complex plane, obtained as the limit of a sequence of maximal average distances between points of the set, the average being furnished by the geometric mean. Polya and Szego (1931), using harmonic, arithmetic, and pth root pth power means instead, extended the definition to three dimensions. All these definitions apply for compact sets in arbitrary metric spaces and so do the corresponding definitions of the Chebichev constant of the set. The notions of capacity also extend for Euclidean spaces at least.

3. *The SMSG geometry program*, by Professor Edwin Moise, Harvard University.

4. *The mathematical training of physics and engineering students*, by Professor R. J. Walker, Cornell University.

5. *Recommendations for teacher training*, by Professor J. G. Kemeny, Dartmouth College.

R. S. PIETERS, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual fall meeting of the Philadelphia Section of the Mathematical Association of America was held on November 26, 1960 at Swarthmore College, Swarthmore, Pennsylvania. Professor W. S. Lawton, Chairman of the Section, presided at the morning and afternoon sessions. There were 70 persons registered in attendance, including 55 members of the Association.

The following officers were elected to serve during the year 1960–61; Chairman, Professor S. S. McNeary, Drexel Institute of Technology; Secretary-Treasurer, Professor F. L. Dennis, Ursinus College; Second member of the Executive Committee, Mr. A. M. Linton, Jr., William Penn Charter School; Third member of the Executive Committee, Professor David Rosen, Swarthmore College. At the business meeting, Professor J. A. Brown reported on the progress of the Committee on Professional Standards.

The following papers were presented:

1. *Some basic concepts in algebraic topology*, by Professor S. L. Gulden, Lehigh University.

Theoretically, a major aim of topology is the identification and classification of all spaces. However, because of the infinite varieties of spaces possible, this goal seems unachievable. A more realistic goal is to restrict one's investigation to a class of spaces having some "relatively nice" properties and to look for some invariants that would give at least a necessary condition for the

identification of these spaces. Such invariants are the homology and cohomology groups of a space. On the class of simplicial complexes, the above invariants do provide an excellent tool for solving many topological problems.

2. *Some nonlinear aspects of differential equations*, by Professor Solomon Lefschetz, Director, Center for Differential Equations RIAS (by invitation of the Executive Committee).

It would, of course, be extremely desirable to be able to solve at least a very general and extensive class of differential equations. However, this is not in the cards and the best thing that one can do is to obtain some idea of the performance of the solutions. Along this line there are two noteworthy topics: (a) the study, in the large following Poincairé, of a system of two first order equations in the plane; (b) the Liapunov stability theory. These two topics are discussed at some length in the present communication.

3. Panel Discussion: *Professional standards for teachers of mathematics in the schools*. Keynoter, Professor Howard Fehr, Teachers College, Columbia University; Moderator, Professor B. H. Bissinger, Lebanon Valley College; Panel, Professor A. E. Filano, West Chester State Teachers College; Mr. K. S. Kalman, Abraham Lincoln High School, Philadelphia; Mr. Joseph Gavin, Olney High School, Philadelphia.

What teachers must know depends upon the subject matter they will teach. What will be taught depends largely on the calibre of scholar that can be attracted to the teaching of high school mathematics. On the assumption that the high school program will eventually be of the standards shown in the SMSG materials, the teachers must have a five-year training period. Entrance to the program should demand four years (9–12) of high school mathematics study as prerequisite. The four year undergraduate program should consist of calculus and analytic geometry (12 s.h.); algebra (polynomial, linear, abstract), 6 s.h.; geometry (affine, euclidean, vector, projective, algebraic), 6 s.h.; probability and statistical inference, 6 s.h.; professionalized subject matter, 6 s.h.; methods of teaching and practice teaching, 6 s.h. The fifth year should include a 3 to 4½ s.h. course in each of the following: (a) higher analysis or function theory; (b) theory of numbers; (c) structures, i.e., theory of sets, topology, or vector spaces; (d) logic or non-euclidean geometries; (e) applications, i.e. mathematical physics, econometrics, game theory, statistical analysis, etc.; and (f) history of mathematics. All this should be accompanied by a seminar in mathematical education.

F. L. DENNIS, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Fall Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on Saturday, December 3, 1960. Professor D. B. Lloyd, Chairman of the Section, presided. One hundred and twenty-three persons were present, including 103 members of the Association. The following papers were presented:

1. *Generalized symmetric random walks*, by Mr. Eugene Albert, University of Virginia.

Define a generalized symmetric random walk as a Markov chain for which: a) for each state E_i, the number of states N_i from which E_i can be reached in one step equals the number of states which can be reached from E_i in one step; b) the one-step transition probabilities from E_i to these N_i states is a constant, for fixed i, to be denoted by a_i. The theorem is proved: The limiting probability of E_i is proportional to $1/a_i$, for any generalized symmetric random walk with a finite number of states.

2. *Preliminary remarks on a distillation problem*, by Mr. H. H. Barnett, Automatic Computations Section, The Martin Company, Baltimore, Maryland.

Periodic provisioning of a component during the operational life of a missile system, in order to reduce to a specified low level the risk of running out of spare components, is a two-stage, multiple-channel flow problem. The first stage is the set of applications of the component; the second stage

is the set of repair areas. For given distributions of failure and repair times, a list of periodic replenishment quantities can be obtained which will satisfy the risk requirements. This is simulated by means of an IBM 709 computer program, using random number generation (Monte Carlo method). Operational life histories are generated until an adequate sample is obtained to make the statistical calculations.

3. *Pursuit games of kind*, by Dr. Rufus Isaacs, Department of Defense, Weapons System Evaluation Group, Washington, D. C.

Pursuit and evasion contests with an essentially two-valued payoff are considered. Under what conditions does the Pursuer P possess a strategy that insures capture of the Evader E despite all opposition? When does E have a strategy guaranteeing escape? The general technique—construction of a surface which separates the space of starting positions into regions of the above two types—is applied to several examples. In the Lifeline and Deadline Games, both players move with fixed speed in a half-plane. Its boundary L is forbidden to E in the latter game, but in the former L is a haven for him: E's reaching L counts as escape. The solutions to the two games are strikingly diverse. In other cases, the motions of P and E are subject to bounded curvature or acceleration. Collision avoidance between moving craft is capable of a modified analysis.

4. *Probability models for measurement with a linear scale*, by Dr. Churchill Eisenhart, Chief, Numerical Analysis Section, U. S. Bureau of Standards.

Let a length of $(m+\delta)w$ be measured with a linear scale having sub-divisions w units apart, where m is some nonnegative integer and $0<\delta<1$. If the scale is positioned at random and the nearest scale divisions at each end of the length "read" correctly in each instance, the sum of n such length determinations will have a binomial distribution with mean $n(m+\delta)w$ and variance $n\delta(1-\delta)$, and standard best-estimate and confidence-interval techniques will be applicable with only slight modifications. Correct estimation of "tenths" corresponds to using a finer mesh, but tenths-estimation errors yield quadrinomial distributions and require modifications in technique.

5. *The training of inservice teachers of mathematics*, by Dr. C. R. Phelps, Program Director for the Academic Year Institutes Program, National Science Foundation.

A discussion of the possible impact of teacher training programs on the schools in the area covered by our section, including present data and future possibilities.

6. *Semantic information*, by Dr. C. J. Maloney, Chief, Bio-mathematics Division, U. S. Army, CmlC Biological Laboratories, Fort Detrick, Maryland.

The theory of information as used in communication is concerned with the development of a procedure for coding messages such that maximum transmission under assigned channel capacity and noise interference is possible. The semantic content of the messages is ignored. Certain applications, including that of information retrieval, require a theory competent to handle the message content. For this purpose two requirements are necessary. Freedom of coding must be restricted to analytical rather than enumerative codes. As words are an enumerative code, messages expressed in words must first be transformed into an analytical code, such as a classification system.

7. *The algebra program in the Soviet Union*, by Professor G. H. Miller, State Teachers College at Towson.

The analysis of the algebra instruction in the U.S.S.R. shows the same traditional topics that we employ in our school systems. During the 6th and 7th years of the Russian schools, the students take the equivalent of a year to a year and a half of algebra. Students graduating from their 10 year and newly revised 11 year school complete the equivalent of our college algebra. Inequalities receive much greater emphasis in the Russian texts than in the American. The teaching of algebra extends over a four and one half year period (or five and one half in the 11 year school) compared to two or three years in our current curriculum.

8. *Evolving patterns in mathematical research*, by Dr. F. J. Weyl, Director of Research, Office of Naval Research. (Invited Address)

An account is rendered of how the scope and substance of mathematical research have been influenced by and have, in turn, influenced the entire supporting apparatus on which its conduct has come to depend in our time. Relevant factors to be analyzed include, aside from the dynamic trends proper to mathematics itself, the new environment created by developments in the rest of science, matters of management and administration, and the general social setting. This sets the stage on which the main protagonists, the universities and colleges, the industrial complexes, and the agencies of the Federal Government, play out their parts in determining the research environment of contemporary mathematics. On the basis of the forces and constraints thus laid bare the attempt is made to describe what can be expected to happen to mathematical research in the course of the next few years ahead.

The meeting concluded with the showing of the color film, *Mathematical Induction*, by Professor L. A. Henkin, University of California, Berkeley.

HERTA T. FREITAG, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.

IOWA, Simpson College, Indianola, April 14, 1961.

KANSAS, Ottawa University, April 15, 1961.

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Aberdeen, Maryland, April 29, 1961.

METROPOLITAN NEW YORK, Fordham University, New York, April 15, 1961.

MICHIGAN

MINNESOTA, St. Cloud State College, May 13, 1961.

MISSOURI, University of Missouri, Columbia, April 22, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA, Oklahoma State University, Stillwater, Spring, 1961.

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA

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TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

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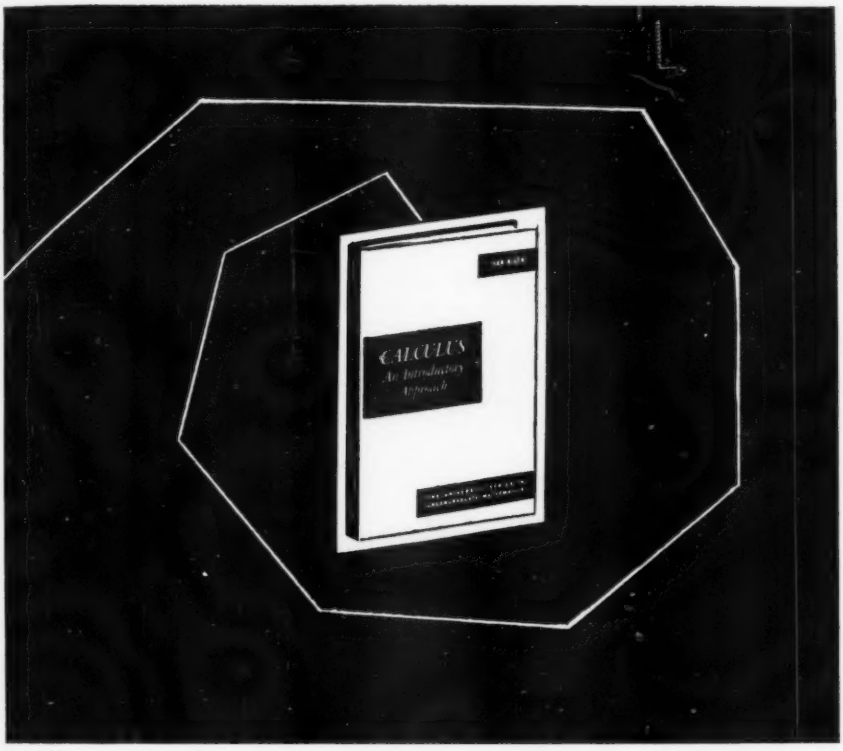

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